The Ganea conjecture and recent developments on Lusternik-Schnirelmann category

Norio Iwase

Introduction

Lusternik-Schnirelmann category (L-S cat for short) is defined by Lusternik and Schnirelmann [48] in 1934 as a numerical homotopy invariant of a manifold M which gives a lower-bound for the number of critical/stationary points of a smooth real-valued function on M. In recent articles, however, L-S category is often defined as the original value minus 1, which is called normalised L-S category. It is just like an initial integer of numeric numbers - 0 or 1. In this article, we call a normalised L-S category a 'cat'. This invariant can also be defined for a general topological space which measures complexity of a given space.

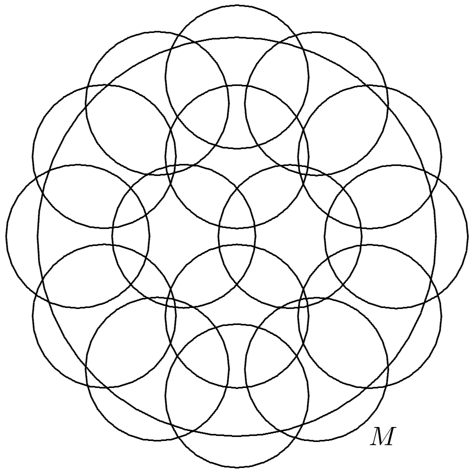

Figure 1

For example, Matumoto [50, 51] and Komatsu [44, 43] use it in knot theory to measure the complexity of knots and links: Under some mild conditions, a knot (or a link) is trivial if and only if the L-S cat of complement of a knot (or a link) is 1. Also, it is used in group theory to measure the complexity of a group with discrete topology. Eilenberg-Ganea showed that L-S cat and geometric dimension (of its classifying space) are the same except for a few cases in 1957. A conjecture says they are always the same. But still at present, the answer is not known. For a given small number, a space with L-S cat equals to the number is classified. A space whose L-S cat is 0 is a contractible space. So a space whose L-S category is 1 must be the simplest non-trivial space and is a co-Hopf space. While a co-Hopf space is simplest, basic problems on co-Hopf spaces like Ganea's conjecture on co-Hopf spaces is not very simple: the conjecture is proved for every prime $p \geq 0$ by Henn [22], Hubbuck-I. [25], but is disproved integrally by I. [29].

In recent years, it has become clearer by Hofer [24], Floer [12] and Rudyak [60] that the Arnold conjecture on symplectic manifolds has a tight connection with L-S cat, by means of (stable) homotopy invariants. Although the product formula on L-S category is verified in rational homotopy category by Hess [23] and Felix-Halperin-Lemaire [9], the original conjecture is solved in negative by I. [28, 30].

In this article, we are following most part of historical descriptions to James [37, 38] and Whitehead [78, 79]. The rational homotopy theory provides strong algebraic tools to study rational L-S cat. Felix Halperin Thomas [10] and Cornea Lupton Oprea Tanré [6] are good references for rational L-S cat. Also [38] and [6] provide general references for L-S cats.

We are also following notations to the majority of authors with some exceptions, e.g., e and c, because they have special meaning[1)] in ordinary homotopy theory. To avoid confusion, we adopt more informative notation, e.g., we use cup[2)] for cup-length and wgt[3)] for toomer invariant and category weight. From now on, we assume that every topological space is a connected CW complex, and hence we can regard a continuous map as a cellular map, up to homotopy.

1. Lusternik-Schnirelmann's 'cats'

Let us recall the relationship between classical cats due to Lusternik and Schnirelmann

1.1. **Geometric 'cats'.** Let M be a differentiable closed manifold. How many critical points does a smooth function on M have at least? We denote by $\mathrm{Crit}(M)$ (or $f\mathrm{Cat}(M)$) the <u>smallest number</u> (or the smallest number minus 1) of such critical points of all smooth function on M. Generally, the Hessian at a critical point can be degenerate, and hence a critical point corresponds to an embedded closed ball (see Takens [72]) in a ball covering of M. — So, how many closed balls are enough to cover (see Figure 1) entire M? We denote by $\mathrm{Ball}(M)$ the <u>least number</u>, which satisfies the following by Takens.

Theorem 1.1 (Takens [72]). $\mathrm{Ball}(M) \le \mathrm{Crit}(M) = f\mathrm{Cat}(M){+}1 \le \dim(M){+}1$.

Another geometric cat $g\mathrm{Cat}$ is introduced by Fox is not a homotopy invariant like fCat.

Definition 1.2 (Fox [13]). *Let X be a topological space. How many closed subsets[4)] are enough to cover entire X? We denote by $g\mathrm{Cat}(X)$ the <u>least number</u> minus 1.*

1.2. **Classical 'cat'.** We may regard that an L-S cat is a homotopy invariant version of Fox's geometric cat. A subset A of a space X is called categorical if the inclusion map $i : A \hookrightarrow X$ is null-homotopic.

Definition 1.3 (Lusternik-Schnirelmann [48]). *How many closed categorical subsets are enough to cover X? We denote by $\mathrm{cat}(X)$ the <u>least number</u> minus 1, which is called <u>L-S cat</u>.*

According to Whitehead [78, 79], Berstein-Ganea [1], the definition of an L-S cat for a manifold M, we may replace 'a closed subset' by 'an NDR closed subset' = 'a closed subset with homotopy extension property'. For each categorical NDR closed subset A, the null-homotopy of the inclusion map $i_A : A \hookrightarrow X$ is extendable to a homotopy deforming the identity of X to a map $r : X \to X$, $r(A) = \{*\}$. By just placing $m{+}1$ such r's for all categorical NDR closed subsets, we obtain a contraction of the $m{+}1$-fold diagonal into a 'fat wedge' - $\prod_m^{m+1} X = \{(x_0, x_1, ..., x_m) | \exists i \text{ s.t. } x_i = *\}$ of $\prod^{m+1} X$. So in this article, we adopt the following widely-accepted definition of <u>L-S cat</u>:

Definition 1.4 (Whitehead [78, 79], Berstein-Ganea [1]). *For a space X, we define*

$$\mathrm{cat}(X) = \mathrm{Min}\left\{ m \ge 0 \,\middle|\, \begin{array}{l} \textit{There is a compression of the } m{+}1\textit{-fold diagonal map } \Delta^{m+1} : \\ X \to \prod^{m+1} X \textit{ into a subspace } \prod_m^{m+1} X \subseteq \prod^{m+1} X. \end{array} \right\}$$

Remark 1.5. *Thus, L-S cat is a numerical invariant ranging over non-negative integers.*

We may say that L-S cat is a homotopy-theoretical version of the geometric cat $g\mathrm{Cat}$. Let us summarise the relationship among these cats:

Theorem 1.6 (L-S [48], Fox [13], Takens [72], Ganea [15], Whitehead [79])**.**

(1) $\mathrm{cat}(M) \le g\mathrm{Cat}(M) \le \mathrm{Ball}(M){-}1 \le \mathrm{Crit}(M){-}1 \le \dim(M)$, *for closed manifold* M

(2) $\mathrm{cat}(X) \le g\mathrm{Cat}(X) \le \dim(X)$ *for a topological space* X.

1.3. **Classical strong 'cat'.** Is L-S cat a unique homotopy-theoretical version of $g\mathrm{Cat}$? Ganea answered 'no' by defining another homotopy invariant called 'strong' cat.

Definition 1.7 (Ganea [15])**.** *Let Y be any space homotopy equivalent to X. The strong cat* $\mathrm{Cat}(X)$ *is defined to be the minimum value of* $g\mathrm{Cat}(Y)$, *where Y ranges over all such spaces:*

$$\mathrm{cat}(X) \le \mathrm{Cat}(X) \le g\mathrm{Cat}(X) \le \dim(X).$$

Ganea also gives the purely homotopy theoretical characterisation of 'strong' cat, which is often called a 'cone-length'.

Definition 1.8. *For any map $h : A \to B$, a mapping cone $C(h)$ is a space obtained from a topological sum $\{*\} \coprod A{\times}[0,1] \coprod B$ by identifying $(a,1) \in A{\times}[0,1]$ with $h(a) \in B$, and $(a,0) \in A{\times}[0,1]$ with $*$. Then, B is canonically identified with a subspace of $C(h)$ through the inclusion $B \hookrightarrow \{*\} \coprod A{\times}[0,1] \coprod B$ and the identification $\{*\} \coprod A{\times}[0,1] \coprod B \to C(h)$.*

Definition 1.9 (Ganea [15])**.** *Let us consider a set of maps $\{h_n : A_n \to Y_n \;\; m{-}1 \ge n \ge 0\}$ satisfying $Y_0 = \{*\}$, $Y_{n+1} = C(h_n) \supset Y_n$ $(m{-}1 \ge n)$ and $Y_m = C(h_{m-1}) \simeq X$ for some $m \ge 0$. The cone length* $\mathrm{Cone}(X)$ *is defined to be the minimum value of the above m, where $\{h_n : A_n \to Y_n \;\; m{-}1 \ge n \ge 0\}$ ranges over all such sets.*

Theorem 1.10 (Ganea [15])**.** *For any space X, the equation* $\mathrm{Cone}(X) = \mathrm{Cat}(X)$ *holds.*

Later in 1990s, Cornea introduced a stronger notion of cone-length by restricting the source space of maps h_n to be an n-fold suspension space.

Definition 1.11 (Cornea [4])**.** *Let us consider a set of maps $\{h_n : \Sigma^n B_n \to Y_n \;\; m{-}1 \ge n \ge 0\}$ satisfying $Y_0 = \{*\}$, $Y_{n+1} = C(h_n) \supset Y_n$ $(m{-}1 \ge n)$ and $Y_m = C(h_{m-1}) \simeq X$ for some $m \ge 0$.* $\mathrm{Cl}(X)$ *is defined to be the minimum value of the above m, where $\{h_n : \Sigma^n B_n \to Y_n \;\; m{-}1 \ge n \ge 0\}$ ranges over all such sets.*

Theorem 1.12 (Cornea [5])**.** *For any space X, we have* $\mathrm{Cl}(X) = \mathrm{Cone}(X)$ $(= \mathrm{Cat}(X))$.

These results suggests that a strong version of L-S cat is essentially unique. So we just call it a strong cat and denote by their original name Cat. We can also define yet another stronger version of cone-length by restricting the source space of maps h_n to be an one-point-sum of spheres of dimension $\ge n$, and obtain a cellular cone-length Cl_S. Although this is also an interesting invariant, we restrict ourselves into the ordinary L-S cat and its family.

Theorem 1.13 (L-S [48], James [37], Takens [72, 73], Ganea [15])**.** *For any space X, we have*

$$\mathrm{Cat}(X){-}1 \le \mathrm{cat}(X) \le \mathrm{Cat}(X) \le g\mathrm{Cat}(X) \le \dim(X).$$

1.4. **Classical weak 'cats'.** The unique strong cat gives an upper-bound for an ordinary L-S cat by constructing a cone decomposition of a space X. There are, of course, weaker but more computable invariants. The following invariants are classically known.

Definition 1.14 (Whitehead [78, 79])**.**

$$w\mathrm{cat}(X) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \bar{\Delta}^{m+1} : X \to \prod^{m+1} X \to \bigwedge^{m+1} X \text{ is null-homotopic}\right\},$$

where $\prod^{m+1} X/\prod^{m+1}_m X = \bigwedge^{m+1} X$*, the* m+1*-fold smash product of* X*.*

Then the following is an immediate consequence from the above definition.

Theorem 1.15 (Whitehead [78, 79])**.** (1) *The inequality* $w\mathrm{cat}(X) \leq \mathrm{cat}(X)$ *holds.*

(2) *Let* h^* *be a multiplicative cohomology theory. If a product of some* m *elements in* $\tilde{h}^*(X)$ *is non-trivial, then we have* $w\mathrm{cat}(X) \geq m$*.*

Outline of the proof: To show (1), we assume $\mathrm{cat}(X) = m$. Since $\Delta^{m+1} : X \to \prod^{m+1} X$ is compressible into $\prod^{m+1}_m X$, the reduced diagonal $\bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X$ is null-homotopic. Thus we have $w\mathrm{cat}(X) \leq m = \mathrm{cat}(X)$. To show (2), we assume $w\mathrm{cat}(X) < m$. By the definition of wcat, the reduced diagonal $\bar{\Delta}^m : X \to \bigwedge^m X$ is null-homotopic. Then the product of any m elements in $\bar{h}^*(X)$ is in the image of the following composition, and hence is trivial.

$$\bar{h}^*(X)\otimes_{h^*}\cdots\otimes_{h^*}\bar{h}^*(X) \longrightarrow \bar{h}^*(X\wedge\cdots\wedge X) \xrightarrow{\bar{\Delta}^{m*}} \bar{h}^*(X)$$

It contradicts to the existence of non-trivial product of m elements. $\square$

Definition 1.16. *For any space* X*, we define a cup-length* $\mathrm{cup}(X)$ *as a numerical invariant.*

(1) *Let* h^* *be a multiplicative cohomology theory.*
$\mathrm{cup}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \forall\{u_0, \cdots, u_m \in \tilde{h}^*(X)\}\; u_0{\cdot}u_1{\cdot}\cdots{\cdot}u_m = 0\right\}$
When h *is an ordinary cohomology theory with coefficient ring* R*, we often abbreviate* $\mathrm{cup}(X;h)$ *by* $\mathrm{cup}(X;R)$*.*

(2) $\mathrm{cup}(X) = \mathrm{Max}\,\{\mathrm{cup}(X;h)\,|\,h$ *is a multiplicative cohomology theory*$\}$

Theorem 1.17. *Let* h^* *be a multiplicative cohomology theory. Then we have*

(1) $\mathrm{cup}(X;h) \;\leq\; \mathrm{cup}(X) \;\leq\; w\mathrm{cat}(X) \;\leq\; \mathrm{cat}(X) \;\leq\; \mathrm{Cat}(X)$.

(2) $\mathrm{cup}(X) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X \text{ is stably null-homotopic}\right\}$.

Proof: The inequality (1) is clear by definition. To show (2), we assume that the right-hand-side of the equality is m. Then the definition of a cup-length immediately implies $\mathrm{cup}(X) \leq m$. So we are left to show $\mathrm{cup}(X) \geq m$: Let $\mathcal{E}_X$ be a multiplicative spectrum defined as

$$\mathcal{E}_X = (S^0) \vee (X) \vee \bigwedge^2(X) \vee \cdots \vee \bigwedge^m(X) \vee \bigwedge^{m+1}(X) \vee \cdots .$$

We define a multiplicative cohomology theory h^* by $h_X(-) = \{(-), \mathcal{E}_X\}$. Let $\iota \in \tilde{h}^*_X(X) = \{(X), \mathcal{E}_X\}$ be the stable class represented by the inclusion of of (X) in $\mathcal{E}_X$. Then the stable class $\iota^m = \bar{\Delta}^{m*}(\iota\otimes\cdots\otimes\iota) \in \tilde{h}^*_X(X) = \{(X), \mathcal{E}_X\}$ is represented by the reduced diagonal $\bar{\Delta}^m : X \to \bigwedge^m X \subset \mathcal{E}_X$, and hence $\bar{\Delta}^m$ is stably trivial. Thus we have $\mathrm{cup}(X) \geq \mathrm{cup}(X;h_X) \geq m$. $\square$

2. Computations of L-S 'cats'

In this section, we observe how we compute L-S category using elementary properties of cats and upper and lower estimates.

2.1. **General properties of L-S 'cats'.**

Fact 2.1. (1) $\mathrm{cat}(X)=0$ *if and only if X is contractible.*

(2) $\mathrm{cat}(X)=1$ *if and only if X is a non-contractible co-Hopf space.*

(3) *If X dominates Y, then we have* $\mathrm{cat}(X)\geq\mathrm{cat}(Y)$.

(4) *(Varadarajan* [77]*, Hardie* [21]*) For a fibration $F\hookrightarrow E\xrightarrow{p}B$, we have* $\mathrm{cat}(E){+}1\leq(\mathrm{cat}(F){+}1)\cdot(\mathrm{cat}(B){+}1)$.

(5) *For the same $F\hookrightarrow E\xrightarrow{p}B$ as above, we have*[5)] $\mathrm{Cat}(E){+}1\leq(\mathrm{Cat}(F){+}1)\cdot(\mathrm{Cat}(B){+}1)$.

(6) *(Fox* [13]*) For any two spaces X and Y, we have* $\mathrm{cat}(X{\times}Y)\leq\mathrm{cat}(X){+}\,\mathrm{cat}(Y)$.

(7) *(Takens* [73]*) For any two spaces X and Y, we have* $\mathrm{Cat}(X{\times}Y)\leq\mathrm{Cat}(X){+}\,\mathrm{Cat}(Y)$.

Theorem 2.2 (Ganea [15])**.** *Let X be $d{-}1$ connected, $d\geq 2$, then we have* $\mathrm{Cat}(X)\leq\dim(X)/d$.

Outline of the proof.[6)] Let us assume that $(d{-}1)$-skeleton of X is one-point-space $\{*\}$. Let X_k be the $((k{+}1)d-1)$-skeleton of X for any $k\geq 0$. Then X_{k+1}/X_k, $k\geq 1$ is a suspension of some space K_k, for dimensional reasons: $X_1\simeq\Sigma K_0$ and $X_{k+1}/X_k\simeq\Sigma K_k$, $k\geq 1$. We remark that K_k is also a suspension space if $k\geq 1$. Since $X_0=\{*\}$, we have $X_1\simeq X_0\cup_{h_0}C(K_0)$, $h_0=*:K_0{\to}\{*\}$. For $k\geq 1$, the homomorphism $\pi_q(X_{k+1},X_k)\to\pi_q(\Sigma K_k)$ is isomorphism if $q<(k{+}2)d{-}1$, and epimorphism if $q=(k{+}2)d{-}1$. Let F_k be the homotopy fibre of the collapsing $f_k:X_{k+1}\to X_{k+1}/X_k$. Then we obtain the following commutative ladder of the exact lows: the exact sequences associated to the fibration $F_k\hookrightarrow X_{k+1}\to\Sigma K_k$ and to the pair (X_{k+1},X_k).

$$\begin{tikzcd}
\pi_{q+1}(X_{k+1},X_k) \arrow[r] \arrow[d] & \pi_q(X_k) \arrow[r] \arrow[d] & \pi_q(X_{k+1}) \arrow[r] \arrow[d, equal] & \pi_q(X_{k+1},X_k) \arrow[d] \\
\pi_{q+1}(\Sigma K_k) \arrow[r] & \pi_q(F_k) \arrow[r] & \pi_q(X_{k+1}) \arrow[r] & \pi_q(\Sigma K_k).
\end{tikzcd}$$

By the four lemma, the canonical inclusion $X_k\hookrightarrow F_k$ is $(k{+}2)d-2$ connected. Since $\dim(K_k)\leq(k{+}2)d-2$, $f_*:[K_k,X_k]\longrightarrow[K_k,F_k]$ is an epimorphism by the theorem of J.H.C. Whitehead, and hence we obtain another epimorphism using a commutative ladder with exact lows similar to the above ladder by replacing S^{q-1} by K_k:

$$[C(K_k),K_k;X_{k+1},X_k]\to[\Sigma K_k,X_{k+1}/X_k]=[\Sigma K_k,\Sigma K_k]$$

Choose a class $[\chi_k]$ in $[C(K_k),K_k;X_{k+1},X_k]$ whose image by the above epimorphism is the identity of ΣK_k. Let $h_k=\chi_k|_{K_k}:K_k\to X_k$. Then $\chi_k$induces a homology equivalence $\tilde{\chi}_k:X_k{\cup_{h_k}}C(K_k)\to X_{k+1}$. Since every space here is 1 connected, we have $X_{k+1}\simeq X_k{\cup_{h_k}}C(K_k)$, $h_k:K_k{\to}X_k$ $(k{\geq}1)$, and hence we have $\mathrm{Cat}(X_m)\leq m$, $m{\geq}0$. If $\dim(X)=nd+r$, $0\leq r<d$ for some n and r, then it follows that $\dim X\leq(n+1)d-1$ and that $\mathrm{Cat}(X)=\mathrm{Cat}(X_n)\leq n\leq\dfrac{\dim(X)}{d}$. □

Ganea has shown further more:

Theorem 2.3 (Ganea [15]). *Let a space X is a $(d-1)$ connected space of* $\dim(X) \leq (k+2)d-3$. *Then the inequality* $\mathrm{cat}(X) \leq k$ *implies* $\mathrm{Cat}(X) \leq k$ *for* $d \geq 2$.

The homotopy set $[\Sigma A, X]$ is a group natural with respect to X. Let $i_t : \Sigma A \hookrightarrow \Sigma A \vee \Sigma A$ be the inclusion into the t-th component and $p_t : \Sigma A \vee \Sigma A \to \Sigma A$ the projection onto t-th component, $t = 1, 2$. Hence $p_{s*} i_t = \delta_{s,t} \cdot 1_{\Sigma A}$, where $\delta s, t$ denotes Kronecker's delta. Then by Ganea [16], the homotopy fibre of the inclusion $\Sigma A \vee \Sigma A \subset \Sigma A \times \Sigma A$ is homotopy equivalent to $\Omega\Sigma A * \Omega\Sigma A$:

Theorem 2.4. *For any A and B, the following gives an exact sequence*

$$1 \to [\Sigma B, \Omega\Sigma A * \Omega\Sigma A] \xrightarrow{[e_1,e_2]_*} [\Sigma B, \Sigma A \vee \Sigma A] \xrightarrow{p_{1*} \times p_{2*}} [\Sigma B, \Sigma A] \times [\Sigma B, \Sigma A] \to 1.$$

with a splitting $i_{1*} \times i_{2*}$, where $[e_1, e_2]$ is the generalised Whitehead product of $e_1 = i_1 \circ \mathrm{ev}$ and $e_2 = i_2 \circ \mathrm{ev}$, and ev *denotes the evaluation map* $\mathrm{e}_1^{\Sigma \mathrm{A}} : \Sigma\Omega\Sigma A \to \Sigma A$.

By pulling this fibration on $\Sigma A \times \Sigma A$ back to ΣA by the diagonal $\Delta : \Sigma A \to \Sigma A \times \Sigma A$, we have a fibration $\Omega\Sigma A * \Omega\Sigma A \xrightarrow{p_1^{\Omega\Sigma A}} \Sigma\Omega\Sigma A \xrightarrow{e_1^{\Sigma A}} \Sigma A$ by Ganea [16], which coincides with Sugawara's Hopf fibration for a Hopf space $\Omega\Sigma A$. Thus we obtain the following split exact sequence:

$$1 \to [\Sigma B, \Omega\Sigma A * \Omega\Sigma A] \xrightarrow{p_1^{\Omega\Sigma A}{}_*} [\Sigma B, \Sigma\Omega\Sigma A] \xrightarrow{e_1^{\Sigma A}{}_*} [\Sigma B, \Sigma A] \to 1,$$

with splitting $\sigma(\Sigma A)_* : [\Sigma B, \Sigma A] \to [\Sigma B, \Sigma\Omega\Sigma A]$ where $\sigma(\Sigma X)$ is given by $\sigma(\Sigma X)(t, x) = (t, \ell_x)$, $\ell_x(u) = (u, x)$. Then for any map $f : \Sigma B \to \Sigma A$, we have $e_1^{\Sigma A} \circ \sigma(\Sigma A) \circ f \simeq f \simeq e_1^{\Sigma A} \circ \Sigma\, \mathrm{ad}\, f = e_1^{\Sigma A} \circ \Sigma\Omega f \circ \sigma(\Sigma B)$, where $\mathrm{ad}(f)$ is the adjoint of f given by $\mathrm{ad}(f)(b) = f \circ \ell_b$.

Definition 2.5 (B-H [2]). *For any map $f : \Sigma B \to \Sigma A$, there is a unique map $g : \Sigma B \to \Omega\Sigma A * \Omega\Sigma A$ up to homotopy, which satisfies $p_1^{\Omega\Sigma A} \circ g \simeq \sigma(\Sigma A) \circ f - \Sigma\Omega f \circ \sigma(\Sigma B)$. We denote such g by $H_1(f)$ which is called a Berstein-Hilton's (1st order) Hopf invariant.*

Remark 2.6. *The original definition of a higher Hopf invariant H_m, $m \geq 1$, is associated with the homotopy-theoretical definition of L-S cat, which implies that for $f : \Sigma B \to X$ with* $\mathrm{cat}(X) \leq m$, *$H_m(f)$ lies in the homotopy set $[C\Sigma B, \Sigma B; \prod^{m+1} X, \prod_m^{m+1} X]$. On the other hand, Definition 2.5 can also be extended to give an alternative definition*[7)] *of a higher Hopf invariant. Although the two definitions give the same invariant by I.* [30] *and Stanley* [67]*, the new definition has some advantage, because we can use the strong properties of an A_∞ structure.*

Theorem 2.7 (B-H [2]). *For any map $f : S^q \to S^r$, the L-S cat of an adjunction space $Q = S^r \cup_f e^{q+1}$ satisfies that* $\mathrm{cat}(Q) = 1$ *iff $H_1(f) = 0$, in other words,* $\mathrm{cat}(Q) = 2$ *iff $H_1(f) \neq 0$.*

2.2. **L-S category of compact Lie groups.** We summarise the results[8)] on compact Lie groups.

Example 2.8. (1) $\mathrm{cat}(\mathrm{T}^r) = \mathrm{cat}(S^1 \times S^1 \times \cdots \times S^1) = r$, $r \geq 1$.

More generally, we have $\mathrm{cat}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_r}) = r$, $n_i \geq 1$, $(1 \leq i \leq r)$

(2) *(Singhof* [64, 65]*) For* $n \geq 1$*, we have*

$\mathrm{cup}(\mathrm{U}(n)) = \mathrm{cat}(\mathrm{U}(n))$ $(= n = \mathrm{cup}(\mathrm{U}(n);\mathbb{Z}))$,

$\mathrm{cup}(\mathrm{SU}(n)) = \mathrm{cat}(\mathrm{SU}(n))$ $(= n{-}1 = \mathrm{cup}(\mathrm{SU}(n);\mathbb{Z}))$.

(3) *(Mimura-I.* [33]*, Mimura-Nishimoto-I.* [34]*)*

$\mathrm{cup}(\mathrm{Spin}(n)) = \mathrm{cat}(\mathrm{Spin}(n)) = \mathrm{Cat}(\mathrm{Spin}(n))$.

(4) *(Schweitzer* [63]*, Fernández Suárez-Gómez Tato-Tanré-Strom* [11]*, Mimura-I.* [33]*)*

$\mathrm{cup}(\mathrm{Sp}(n)) = \mathrm{cat}(\mathrm{Sp}(n)) = \mathrm{Cat}(\mathrm{Sp}(n))$ $(= 2n{-}1)$, $n{\leq}3$.

(5) *(Singhof* [64]*, James* [37]*, Mimura-Nishimoto-I.* [35]*)* $\mathrm{cat}(\mathrm{G}_2) = 4$, $\mathrm{cat}(\mathrm{PSp}(2))$[9)] $= 8$.

(6) *(James-Singhof* [39]*) We know* $\mathrm{SO}(2){\approx}S^1, \mathrm{SO}(3){\approx}\mathbb{R}\mathrm{P}^3, \mathrm{SO}(4){\approx}\mathbb{R}\mathrm{P}^3{\times}S^3$*. For* $\mathrm{SO}(n)$*,* $n{\leq}5$*, we have* $\mathrm{cup}(\mathrm{SO}(5)) = \mathrm{cat}(\mathrm{SO}(5)) = \mathrm{Cat}(\mathrm{SO}(5))$ $(= 8 = \mathrm{cup}(\mathrm{SO}(5);\mathbb{Z}/2\mathbb{Z}))$.

(7) *(Mimura-Nishimoto-I.* [35]*) For* $n = p^r$ *with* $r \geq 1$ *and* p *a prime, we have* $\mathrm{wgt}(\mathrm{PU}(n))$[10)] $= \mathrm{cat}(\mathrm{PU}(n)) = \mathrm{Cat}(\mathrm{PU}(n))$ $(= 3n{-}3)$.

(8) *(Mimura-Nishimoto-I.* [35]*) For* n *with* $6 \leq n \leq 9$*, we have* $\mathrm{cup}(\mathrm{SO}(n)) = \mathrm{cat}(\mathrm{SO}(n)) = \mathrm{Cat}(\mathrm{SO}(n))$ $(= \mathrm{cup}(\mathrm{SO}(n);\mathbb{Z}/2\mathbb{Z}))$.

(9) *(Mimura-Nishimoto-I.* [35]*) We know* $\mathrm{PO}(2){\approx}S^1, \mathrm{PO}(4){\approx}\mathbb{R}\mathrm{P}^3{\times}\mathbb{R}\mathrm{P}^3$ *and* $\mathrm{PO}(6) = \mathrm{PU}(4)$ *whose L-S cats are* 1*,* 6 *and* 9*. Thus the first non-trivial case is* $\mathrm{PO}(8)$*: We have* $\mathrm{cup}(\mathrm{PO}(8)) = \mathrm{cat}(\mathrm{PO}(8)) = \mathrm{Cat}(\mathrm{PO}(8)) = \mathrm{cup}(\mathrm{PO}(8);\mathbb{Z}/2\mathbb{Z})$.

2.3. **Problems related to L-S 'cats' and its recent developments.** Some classical problems on L-S category is listed on "Open Problems in Topology" [53]. The following is one of them.

Problem 2.9 (Problem 643 of [53])**.** *Is the L-S cat of a closed manifold bigger than that for once-punctured submanifold just by* 1*?*

The affirmative answer to the following problem is the Arnold conjecture.

Problem 2.10 (Arnold (p.66 of [3]))**.** *Let* (M,ω) *be a symplectic manifold and* $\mathrm{Fix}(\phi)$ *be the number of fixed points of a symplectomorphism* ϕ *on* M*. Is it always true that* $\mathrm{Fix}(\phi) \geq \mathrm{Crit}(M)$*?*

The affirmative answer to the following problem is the Ganea conjecture.

Problem 2.11 (Problem 2 of Ganea [17], Problem 642 of [53])**.** *Does the equality* $\mathrm{cat}(X{\times}S^n) = \mathrm{cat}(X){+}1$ *hold for any* X *and any* $n \geq 1$*?*

The next problem is also related to the Ganea conjecture.

Problem 2.12 (Problem 4 of Ganea [17])**.** *Describe L-S cat of a sphere-bundle over a sphere in terms of homotopy invariants of the characteristic map of the bundle.*

There is a list of some classical results and their improvements.

Theorem 2.13 (Singhof [66], Rudyak [58])**.** *If a closed manifold* M *satisfies an inequality* $m \geq \dfrac{d{+}1}{2}$ *between the dimension* $\dim(M) = d$ *and the L-S cat* $\mathrm{cat}(M) = m$*, then the Ganea conjecture holds for such* M *and any* $n \geq 1$*.*

Theorem 2.14 (Hofer [24], Floer [12])**.** *Let* (M,ω) *be a symplectic manifold. Then we have* $\mathrm{Fix}(\phi) \geq \mathrm{cup}(M) + 1$ *for any symplectomorphism* ϕ *on* M*.*

Theorem 2.15 (Hess [23]). *The rational version of the Ganea conjecture is true.*

In the last 10 years, several results have been developed, which bring us new aspects in this area mainly by category-weight techniques and by higher-Hopf invariants on A_∞ structures.

Theorem 2.16 (Liu-Tian [47], Fukaya-Ono [14]). *For any symplectic manifold (M,ω), we have* $\mathrm{Fix}(\phi) \geq \mathrm{Crit}(M)$*, if each fixed point of a Hamiltonian diffeomorphism ϕ on M is non-degenerate.*

Theorem 2.17 (I. [28]). *There are counter-examples to the Ganea conjecture as* 1 *connected finite complexes.*

Theorem 2.18 (Rudyak [60], Oprea-Rudyak [56]). *Let (M,ω) be a symplectic manifold with $\omega|_{\pi_2(M)} = 0$ (or simply $\pi_2(M) = 0$). Then we have* $\mathrm{Fix}(\phi) \geq \mathrm{Crit}(M)$ *for any symplectomorphism ϕ on M.*

Theorem 2.19 (I. [30]). *There exists a* 1 *connected closed manifold which does not satisfies the Ganea conjecture.*

Theorem 2.20 (I. [30, 31], Lambrechts-Stanley-Vandembroucq [49]). *There exists a* 1 *connected closed manifold whose L-S cat is the same as its once-punctured submanifold.*

Theorem 2.21 (I. [31]). *L-S cat of the total space of a sphere-bundle over a sphere is classified by the higher Hopf invariants of the characteristic map of the bundle. We can observe that there are many counter-examples to the Ganea conjecture.*

Theorem 2.22 (Oprea-Rudyak [57]). *A closed* 3*-manifold satisfies the Ganea conjecture.*

3. A_∞-structure and L-S 'cat'

Higher homotopy associativity - A_m structure - is the author's original working area, especially on an A_m structure of a Hopf space and an A_m structure of a map between A_m spaces - spaces with A_m structures. Just after a lecture of Ioan James at Aberdeen University in 1997, the author noticed that a computation of the L-S cat of Sp(2) can be simplified significantly by an unstable Hopf invariant which can be defined over a projective space associated to an A_∞ structure of ΩSp(2). The idea that an unstable Hopf invariant over an A_∞ structure of ΩX is effective to determine L-S cat of X forced the author to consider it for a couple of months. What was obtained is that a combination of an A_∞ structure and a homology decomposition visualises how a higher Hopf invariant controls the L-S cat under suitable conditions (see §4.4). At about the same period, Yuli Rudyak was working on Arnold conjecture using his category weight - a homotopy-theoretical version of Fadell-Husseini's category weight - which can also be regarded as an invariant using A_∞ structure (see §4.1).

3.1. A_m-structure.

Definition 3.1 (Stasheff [68], I. [26, 27], Mimura [54]). *An A_m structure on a space X is a set of triples $\{(E^k(X), B^k(X), p^k_X) \mid 0{\leq}k{\leq}m\}$ satisfying the following three conditions (1) through (3). A space with an A_m structure is called an A_m space.*

(1) *$B^1(X) = \{*\}$, $E^1(X) = X$ and p^1_X is a trivial map.*

(2) *Since* $E^k(X)$ *is contractible in* $E^{k+1}(X)$ *for* $k < m$, $E^\infty(X) = \bigcup E^k(X)$ *is contractible.*

(3) $p^k_X : E^k(X) \to B^k(X)$ $(k \leq m)$ *is a quasi-fibration with a fibre* $X = E^1(X) \subseteq E^k(X)$.

where we call $p : E \to B$ *a quasi-fibration with a fibre* $F \subseteq E$ *if* $p(F) = \{*\}$ *and* $p_* : \pi_*(E, F) \to \pi_*(B)$ *induces an isomorphism in every dimension.*

Theorem 3.2 (Stasheff [68]). *If a space* X *has an* A_m *structure* $\{B^{k+1}(X) \,|\, m \geq k \geq 0\}$, *then* X *is homotopy equivalent to an* A_m *space* $\tilde{X}$ *with standard* A_m *structure, which is associated to an* A_m *form*[11)] *with higher degeneracy conditions. There is a sequence of maps from* $\tilde{B}^{k+1}(\tilde{X})$ *to* $B^{k+1}(\tilde{X})$ *commutes with the inclusions* $B^{k+1}(\tilde{X}) \hookrightarrow B^{k+2}(\tilde{X})$ *and* $\tilde{B}^{k+1}(\tilde{X}) \hookrightarrow \tilde{B}^{k+2}(\tilde{X})$.

Corollary 3.3. *For any space* X, *an* A_∞ *structure* $\{B^{k+1}(\tilde{\Omega}X) \,|\, k \geq 0\}$ *of a loop space* $\tilde{\Omega}X$ *satisfies* $B^\infty(\tilde{\Omega}X) \simeq X$.

Example 3.4.

(1) $\tilde{B}^{k+1}(S^0) = \mathbb{R}\mathrm{P}^k$, $\tilde{B}^{k+1}(S^1) = \mathbb{C}\mathrm{P}^k$, $\tilde{B}^{k+1}(S^3) = \mathbb{H}\mathrm{P}^k$, $0 \leq k \leq \infty$.

(2) $\tilde{B}^1(S^7) = *$, $\tilde{B}^2(S^7) = S^8$, $\tilde{B}^3(S^7) = \mathcal{O}P^2$ *(Cayley plane).*

Remark 3.5. *The indexing of these examples suggests that* $\tilde{B}^{k+1}(\tilde{\Omega}X)$ *can be abbreviated as* $P^k(\Omega X)$. *In this article, we adopt the latter notation hereafter.*

3.2. **Projective spaces and L-S 'cats'.** The definition of a projective space implies

Theorem 3.6. $\mathrm{Cat}(P^m(\Omega X)) \leq m$, *and hence* $\mathrm{cat}(P^m(\Omega X)) \leq m$.

Theorem 3.7 (Cornea [4]). *Let* $\mathrm{cat}(X) = m$. *Then we have* $\mathrm{cat}(P^i(\Omega X)) = i$ *if* $i \leq m$ *and* $\mathrm{cat}(P^i(\Omega X)) = m$ *if* $i \geq m$.

Theorem 3.8 (Ganea [15], Gilbert [18], I. [28], Sakai [62]). $\mathrm{cat}(X) \leq m$ *if and only if* $e^X_m : P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X$ *has a right homotopy inversion.*

$$\mathrm{cat}(X) = \mathrm{Min}\{m \geq 0 \,|\, \exists \sigma : X \to P^m(\Omega X) \; s.t. \; e^X_m \circ \sigma \sim 1_X\}.$$

Theorem 3.9 (I. [28]). $\mathrm{cat}(X \times Y) = \mathrm{cat}(X) + \mathrm{cat}(Y)$ *if and only if* $\bigcup_{i+j<m+n} P^i(\Omega X) \times P^j(\Omega Y) \hookrightarrow P^\infty(\Omega X) \times P^\infty(\Omega Y) \simeq X \times Y$ *has* <u>*no*</u> *right homotopy inversion.*

Outline of the proof. Firstly we obtain $\mathrm{Cat}(\bigcup_{i+j<k} P^i(\Omega X) \times P^j(\Omega Y)) < k$ by induction on k. Assume that $\bigcup_{i+j<m+n} P^i(\Omega X) \times P^j(\Omega Y) \hookrightarrow P^\infty(\Omega X) \times P^\infty(\Omega Y) \simeq X \times Y$ has a right homotopy inversion: Then $X \times Y$ is dominated by $\bigcup_{i+j<m+n} P^i(\Omega X) \times P^j(\Omega Y)$ whose L-S cat is less than $m + n$, and hence we have $\mathrm{cat}(X \times Y) < m + n = \mathrm{cat}(X) + \mathrm{cat}(Y)$.

Conversely, assume that $\mathrm{cat}(X \times Y) < \mathrm{cat}(X) + \mathrm{cat}(Y)$: Then by Theorem 3.8 implies that $X \times Y$ is dominated by $P^{m+n-1}(\Omega(X \times Y))$, which is the standard A_∞ structure of $\Omega X \times \Omega Y$. Alternatively, the spaces $\bigcup_{i+j \leq k} P^i(\Omega X) \times P^j(\Omega Y)$ also gives a non-standard A_∞ structure of $\Omega X \times \Omega Y$. Then by Theorem 3.2, the standard map

$$e^{X \times Y}_k : P^k(\Omega(X \times Y)) \hookrightarrow P^\infty(\Omega X) \times P^\infty(\Omega Y) \simeq X \times Y$$

goes through the inclusion $\bigcup_{i+j \leq k} P^i(\Omega X) \times P^j(\Omega Y) \hookrightarrow P^\infty(\Omega X) \times P^\infty(\Omega Y)$. Thus $X \times Y$ is dominated also by the subspace $\bigcup_{i+j \leq m+n-1} P^i(\Omega X) \times P^j(\Omega Y) \subseteq P^\infty(\Omega X) \times P^\infty(\Omega Y)$. $\square$

Corollary 3.10 (I. [28]). $\mathrm{cat}(X{\times}S^n) = \mathrm{cat}(X){+}1$ *if and only if* $P^m(\Omega X){\times}\{*\}{\cup}P^{m-1}(\Omega X){\times}S^n \hookrightarrow P^\infty(\Omega X){\times}S^n \simeq X{\times}S^n$ *has a right homotopy inversion, where* $\mathrm{cat}(X) = m \geq 1$.

Outline of the proof. We know $P^m(\Omega X){\times}\{*\}{\cup}P^{m-1}(\Omega X){\times}P^1(\Omega S^n){\cup}\cdots{\cup}\{*\}{\times}P^{m-1}(\Omega S^n) \subseteq P^m(\Omega X){\times}\{*\}{\cup}P^{m-1}(\Omega X){\times}P^\infty(\Omega S^n) \subset P^\infty(\Omega X){\times}P^\infty(\Omega S^n)$. If $\mathrm{cat}(X{\times}S^n) = \mathrm{cat}(X)$, then by Theorem 3.9, $P^m(\Omega X){\times}\{*\} \cup P^{m-1}(\Omega X){\times}P^\infty(\Omega S^n) \hookrightarrow P^\infty(\Omega X){\times}P^\infty(\Omega S^n) \simeq X{\times}S^n$ has a right homotopy inversion. Thus so does $P^m(\Omega X){\times}\{*\}{\cup}P^{m-1}(\Omega X){\times}S^n \hookrightarrow P^\infty(\Omega X){\times}S^n \simeq X{\times}S^n$. The converse is clear by $\mathrm{Cat}(P^m(\Omega X){\times}\{*\}{\cup}P^{m-1}(\Omega X){\times}S^n) \leq m$ □

4. Two paths to climb up

There are two kinds of paths to climb up a mountain of a L-S category problem: One is a wide and stable path made of nice manifolds, where you should use a strong computable invariant as an axe to proceed to make the path. It seems that Rudyak's method comes along this idea. However if you consider some critical problems like the Ganea conjecture, you should find out another path, which must be narrow and unstable, and is made of cellular complex linked by necessary and sufficient conditions, where you should use a sharp theoretical invariant as a knife.

4.1. **Using computable invariants in a stable situation.**

Definition 4.1. *We introduce here the Toomer invariant and its variants.*

(1) *Let h be a multiplicative cohomology theory.*
   (a) $\mathrm{wgt}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, (e^X_m)_* : h^*(X) \to h^*(P^m(\Omega X)) \text{ is a monomorphism}\right\}$
   (b) $\mathrm{Mwgt}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \begin{array}{l}(e^X_m)_* : h^*(X) \to h^*(P^m(\Omega X)) \text{ is a split mono of}\\ \text{unstable } h^*h\text{-modules}\end{array}\right\}$
   (c) $\mathrm{Awgt}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \begin{array}{l}(e^X_m)_* : h^*(X) \to h^*(P^m(\Omega X)) \text{ is a split mono of}\\ \text{unstable } h^*h\text{-algebras}\end{array}\right\}$

(2) (a) $\mathrm{wgt}(X) = \mathrm{Max}\,\{\mathrm{wgt}(X;h) \mid h \text{ is a multiplicative cohomology theory}\}$
   (b) $\mathrm{Mwgt}(X) = \mathrm{Max}\,\{\mathrm{Mwgt}(X;h) \mid h \text{ is a multiplicative cohomology theory}\}$
   (c) $\mathrm{Awgt}(X) = \mathrm{Max}\,\{\mathrm{Awgt}(X;h) \mid h \text{ is a multiplicative cohomology theory}\}$

(3) (a) $\mathrm{wgt}(X;p) = \mathrm{Max}\,\{\mathrm{wgt}(X;h) \mid h \text{ is a multiplicative } p\text{-local cohomology theory}\}$
   (b) $\mathrm{Mwgt}(X;p) = \mathrm{Max}\,\{\mathrm{Mwgt}(X;h) \mid h \text{ is a multiplicative } p\text{-local cohomology theory}\}$
   (c) $\mathrm{Awgt}(X;p) = \mathrm{Max}\,\{\mathrm{Awgt}(X;h) \mid h \text{ is a multiplicative } p\text{-local cohomology theory}\}$

When h is an ordinary cohomology theory with coefficients in R, we often abbreviate $\mathrm{wgt}(X;h)$, $\mathrm{Mwgt}(X;h)$ and $\mathrm{Awgt}(X;h)$ by $\mathrm{wgt}(X;R)$, $\mathrm{Mwgt}(X;R)$ and $\mathrm{Awgt}(X;R)$, respectively.

Theorem 4.2. $\mathrm{cup}(X;h) \leq \mathrm{wgt}(X;h) \leq \mathrm{Mwgt}(X;h) \leq \mathrm{Awgt}(X;h) \leq \mathrm{cat}(X)$.

Independently, Rudyak and Strom gave a homotopy-theoretical version of Fadell-Husseini's category weight (see [8]) to obtain an effective lower bound for an L-S cat.

Definition 4.3 (Rudyak [58, 59], Strom [71]). *For $u \in \tilde{h}^*(X)$, we have*

$$\mathrm{wgt}(u;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, (e^X_m)_*(u) \neq 0\right\} \text{ where } h \text{ is a multiplicative cohomology theory.}$$

Theorem 4.4 (Rudyak [58, 59], Strom [71]). *Let h be a multiplicative cohomology theory.*

(1) $\mathrm{wgt}(u{+}v;h) \geq \mathrm{Min}\{\mathrm{wgt}(u;h), \mathrm{wgt}(v;h)\}$.

(2) $\mathrm{wgt}(uv;h) \geq \mathrm{wgt}(u;h) + \mathrm{wgt}(v;h)$.

(3) $\mathrm{wgt}(X;h) = \mathrm{Max}\{\mathrm{wgt}(u;h) \mid u \in \tilde{h}^*(X)\}$.

Definition 4.5. *Let $\{(E_r^{**}(X;h), d_r) \mid r \geq 1\}$ be the spectral sequence of Rothenberg-Steenrod type associated with the filtration of $X \simeq P^\infty(\Omega X)$ given by $\{P^m(\Omega X) \mid m \geq 0\}$.*

Theorem 4.6 (Whitehead [78], Ginsburg [19], McCleary [52])**.** *Let X be 1 connected.*

(1) *If $h^*(\Omega X)$ is free over h^*, we have $E_2^{**}(X;h) \cong \mathrm{Cotor}^{*,*}_{h^*(\Omega X)}(h^*, h^*)$*

(2) *$d_r : E_r^{s,t}(X;h) \to E_r^{s+r,t-r+1}(X;h)$ and $H(E_r^{*,*}(X;h), d_r) \cong E_{r+1}^{*,*}(X;h)$*

(3) $E_\infty^{*,*}(X;h) \cong E_0 h^*(X), \qquad E_\infty^{s,t}(X;h) \cong F_s h^{s+t}(X)/F_{s+1}h^{s+t}(X)$

$$F_m h^n(X) = \mathrm{Ker}\left\{(e_m^X)_* : h^n(X) \to h^n(P^m(\Omega X))\right\}$$

(4) *(Whitehead) If $r > \mathrm{cat}(X)$, then we have $E_r^{s,t}(X;h) \cong E_\infty^{s,t}(X;h)$.*

(5) *(Ginsburg) If $s > \mathrm{cat}(X)$, then we have $E_\infty^{s,t}(X;h) = 0$.*

Remark 4.7. *For any $[u]\ (\neq 0) \in E_\infty^{s,*}(X;h)$, $(u \in \tilde{h}^*(X))$, $\mathrm{wgt}(u;h) = s$.*

Example 4.8. (1) $\mathrm{wgt}(L^n(p)) = \mathrm{cat}(L^n(p)) = \dim(L^n(p)) = n$ *for all $p > 1$.*

(2) *If a symplectic manifold M^{2n} satisfies $\pi_2(M){=}0$, we have* $\mathrm{wgt}(M) = \mathrm{cat}(M) = 2n$.

(3) $\mathrm{wgt}(\mathrm{Sp}(2); \mathbb{Z}/2) = 2 < 3 = \mathrm{Mwgt}(\mathrm{Sp}(2); \mathbb{Z}/2) = \mathrm{cat}(\mathrm{Sp}(2))$.

4.2. **New stable 'cats'.** Quite recently, some new stable cats are introduced to give a better lower bound for L-S cat than cup. One is computable and the other is theoretical.

Definition 4.9 (Rudyak [59])**.**

$$r\mathrm{cat}(X)^{12)} = \mathrm{Min}\{m{\geq}0 \mid \exists \sigma \in \{X, P^m(\Omega X)\}\ e_m^X{\circ}\sigma \sim 1_X\ \textit{(stably)}\}.$$

Definition 4.10 (Vandembroucq [76])**.**

$$\mathrm{Qcat}(X) = \mathrm{Min}\{m{\geq}0 \mid \exists \sigma : X{\to}(QP)^m(\Omega X)\ s.t.\ (Qe)_m^X{\circ}\sigma \sim 1_X\}$$

where the fibration $Q(E^{m+1}(\Omega X)) \longrightarrow (QP)^m(\Omega X) \xrightarrow{(Qe)_m^X} X$ is obtained by the fibrewise stabilisation of the fibration $E^{m+1}(\Omega X) \longrightarrow P^m(\Omega X) \xrightarrow{e_m^X} X$.

For a symplectic manifold (M,ω), Rudyak shows that $r\mathrm{cat}(M)$ and $\dim M$ give the lower and upper bound for both $\mathrm{Fix}\, M$ and $\mathrm{Crit}(M)$ and that $r\mathrm{cat}(M) = \dim M$ under a suitable condition.

Theorem 4.11 (Rudyak [59, 60], Vandembroucq [76])**.**

(1) $r\mathrm{cat}(X) \leq \mathrm{cat}(X), \quad r\mathrm{cat}(X{\times}S^n) = r\mathrm{cat}(X) + 1,\ n \geq 1$.

(2) $\mathrm{Qcat}(X) \leq \mathrm{cat}(X), \quad \mathrm{Qcat}(X{\times}S^n) = \mathrm{Qcat}(X) + 1,\ n \geq 1$.

(3) *For a rational space X_0, we have* $\mathrm{wgt}(X_0) = r\mathrm{cat}(X_0) \leq \mathrm{Qcat}(X_0) = \mathrm{cat}(X_0)$.

The relationship among these stable cats and algebraic invariants are given as follows:

Theorem 4.12. $\mathrm{cup}(X) \leq \mathrm{wgt}(X) = r\mathrm{cat}(X) \leq \mathrm{Mwgt}(X) \leq \mathrm{Awgt}(X) \leq \mathrm{cat}(X)$.

Just theoretically speaking, as lower bounds for $\mathrm{cat}(X)$, $\mathrm{wgt}(X;h)$ is better than $\mathrm{cup}(X;h)$, $\mathrm{Mwgt}(X;h)$ is better than $\mathrm{wgt}(X;h)$, and $\mathrm{Awgt}(X;h)$ is better than $\mathrm{Mwgt}(X;h)$.

4.3. **Higher Hopf invariants and L-S 'cats'.** Algebraic invariants such as introduced in the earlier sections, can be defined through the stable homotopy theory, which means that it can hardly be effective in the unstable situation like the study of the Ganea conjecture. Instead we can use the following higher (un)stable Hopf invariants. Berstein-Hilton [2] defines a higher Hopf invariant for an element of homotopy groups with coefficients in R:

$$H^s_m : \pi_n(X;R) \to \pi_{n+1}(\textstyle\prod^m X, \prod^m_{m-1} X; R), \quad n{\geq}2,\ m{\geq}1$$

which gives a criterion to determine a cat for a space with two cells other than the base point. We can give an alternative definition of the higher Hopf invariant using the A_∞ structure of ΩX:

Definition 4.13 (I. [30], Stanley [67])**.** *Let $\sigma : X \to P^m(\Omega X)$ be the structure map for $\mathrm{cat}(X) \leq m$ given by Theorem 3.8. For a given map $f : \Sigma V \to X$, Figure 2 below is commutative up to homotopy except for the dotted arrows and the center square, since $e^X_m{\circ}\Sigma\,\mathrm{ad}(f) = \mathrm{ev}{\circ}\Sigma\,\mathrm{ad}(f) = f = 1_X{\circ}f = e^X_m{\circ}\sigma{\circ}f$.*

$$\begin{tikzcd}
 & \Sigma V \arrow[d, "f"] \arrow[dddl, dotted, "H^\sigma_m(f)"'] \arrow[dddr, dotted] \arrow[ddr, "\Sigma\,\mathrm{ad}(f)"] \arrow[dddrr, "f"] & & \\
 & X \arrow[ddr, "\sigma"'] & & \\
 & & \Sigma\Omega X \arrow[d, hook] \arrow[dr, "\mathrm{ev}"] & \\
\tilde{E}^{m+1}(\Omega X) \arrow[rr, "p^{\Omega X}_{m+1}"'] & & P^m(\Omega X) \arrow[r, "e^X_m"'] & X.
\end{tikzcd}$$

Figure 2

The difference between $\sigma{\circ}f$ and $\Sigma\,\mathrm{ad}(f)$ is given by a map $d^\sigma_m(f) = \sigma{\circ}f - \Sigma\,\mathrm{ad}(f)$ whose lift is denoted by $H^\sigma_m(f) \in [\Sigma V, \tilde{E}^{m+1}(\Omega X)] \cong [\Sigma V, \Sigma^m(\bigwedge^{m+1}\Omega X)]$: We have $p^{\Omega X}_{m+1}{\circ}H^\sigma_m(f) = d^\sigma_m(f)$. We also define $\mathcal{H}^\sigma_m(f){=}\Sigma^\infty H^\sigma_m(f) \in \{\Sigma^{1-m}V, \bigwedge^{m+1}\Omega X\}$ the stable set from $\Sigma^{1-m}V$ to $\bigwedge^{m+1}\Omega X$.

Theorem 4.14 (I. [30])**.** *If V is a co-Hopf space, then for each σ, H^σ_m is a homomorphism.*

Outline of the proof. Let $\mathrm{ad}(f), \mathrm{ad}(g) : V \to \Omega X$ be the adjoints of maps $f, g : \Sigma V \to X$:

$$\Sigma\,\mathrm{ad}(f) : \Sigma V \to \Sigma\Omega X, \quad \Sigma\,\mathrm{ad}(g) : \Sigma V \to \Sigma\Omega X, \quad \Sigma\,\mathrm{ad}(f{+_S}g) : \Sigma V \to \Sigma\Omega X,$$

where $+_S$ denotes the multiplication determined by the suspension structure. Let us recall that ΣV has two co-Hopf structures: One is derived from the suspension structure and the other from the co-Hopf structure of V which we denote by $+_V$. Then we have $\mathrm{ad}(f{+_S}g) \sim \mathrm{ad}(f{+_V}g) \sim \mathrm{ad}(f){+_V}\,\mathrm{ad}(g)$, and hence by taking suspension, we have

$$\Sigma\,\mathrm{ad}(f{+_S}g) \sim \Sigma(\mathrm{ad}(f){+_V}\,\mathrm{ad}(g)) \sim \Sigma\,\mathrm{ad}(f){+_V}\Sigma\,\mathrm{ad}(g)) \sim \Sigma\,\mathrm{ad}(f){+_S}\Sigma\,\mathrm{ad}(g).$$

Thus we have $H^\sigma_m(f+g) \sim H^\sigma_m(f) + H^\sigma_m(g)$ by the definition of the higher Hopf invariant. $\square$

Proposition 4.15 (I. [30])**.** *For any σ, we have $H^\sigma_m(f{\circ}(\Sigma g)) = H^\sigma_m(f){\circ}(\Sigma g)$.*

Assume there is an A_m space G with higher degeneracy conditions. Then for $X = P^m(G)$, the standard structure map $\sigma : P^m(G) \to P^m(\Omega P^m(G))$ for $\mathrm{cat}(X) \leq m$, we have an higher Hopf invariant $H^\sigma_m : [\Sigma V, P^m(G)] \to [\Sigma V, \tilde{E}^{m+1}(\Omega P^m(G))]$, where $\tilde{E}^{m+1}(\Omega P^m(G))$ is the $m+1$ fold join of $\Omega P^m(G)$. As the ordinary Hopf invariant H_1 detects a Hopf structure, this higher Hopf invariants detects higher homotopy associativity:

Theorem 4.16 (I. [30])**.** *An $n{-}1$ sphere S^{n-1} for $n = 1$, 2, 4 or 8 is a Hopf space with a multiplication inherited from the (non-associative) division algebra $\mathbb{R}^n$. Then we can easily see that the existence of a higher Hopf invariant one for $H_m : \pi_{n(m+1)-1}(P^m(S^{n-1})) \to \mathbb{Z}$ is the necessary and sufficient condition for the existence of an A_{m+1} structure on S^{n-1}. We remark that such an element exists except for the case when $n = 8$ and $m \geq 2$.*

Definition 4.17 (I. [30])**.** *Let X be a space with $\mathrm{cat}(X) = m$. An (un)stable higher Hopf invariant is defined as the following subset of the (un)stable homotopy sets from ΣV to $\tilde{E}^{m+1}(\Omega X)$:*

$$\begin{cases} H_m(\alpha) = \{H^\sigma_m(\alpha) \,|\, \sigma \text{ is a structure of } \mathrm{cat}(X) = m\} \subseteq [\Sigma V, \tilde{E}^{m+1}(\Omega X)], \\ \mathcal{H}_m(\alpha) = \{\mathcal{H}^\sigma_m(\alpha) \,|\, \sigma \text{ is a structure of } \mathrm{cat}(X) = m\} \subseteq \{\Sigma V, \tilde{E}^{m+1}(\Omega X)\}. \end{cases}$$

Proposition 4.18 (I. [30])**.** *Let X be a $d{-}1$ connected space, $d \geq 2$. If $d{\cdot}\mathrm{cat}(X){+}d{-}2 \geq \dim(X)$, then σ is unique up to homotopy.*

Example 4.19. *$X = S^n, \mathbb{C}\mathrm{P}^n, \mathbb{H}\mathrm{P}^n$, $n \geq 1$, satisfies the above conditions. Hence in this case, we can identify the higher Hopf invariant with its unique element.*

4.4. **Using a necessary and sufficient condition in an unstable situation.**

To keep a required condition to be necessary and sufficient as general as possible, we use a homology decomposition.

Definition 4.20 (Homology decomposition)**.** *Let X be a simply connected space with a cone-decomposition $\{S_i(X) \xrightarrow{f_i} X_i \hookrightarrow X_{i+1}\}$. We call it a homology decomposition if for every i, $S_i(X)$ is a Moore space of type $(H_{i+1}(X), i)$.*

For a given 1 connected space X, we fix a homology decomposition $\{S_i(X) \xrightarrow{f_i} X_i \hookrightarrow X_{i+1}\}$.

Proposition 4.21. *For any $i \geq 1$, we have $\mathrm{cat}(X_i) \leq \mathrm{cat}(X_{i+1})$.*

The following three results shows that we can control the increase of the L-S cat by the higher Hopf invariants using the homology decomposition, under some mild conditions. Thus a higher Hopf invariant plays the crucial role for the Ganea conjecture under such conditions.

Theorem 4.22 (I. [30])**.** *Let $\mathrm{cat}(X_i) = m$ and $i \geq 1$. Then we have*

(1) *$H_m(f_i) \ni 0$ implies $\mathrm{cat}(X_{i+1}) = \mathrm{cat}(X_i)$.*
(2) *Let $\mathrm{cat}(X_{i+1}) = m{+}1$. Then $\Sigma^n_* H_m(f_i) \ni 0$ implies $\mathrm{cat}(X_{i+1}{\times}S^n) = \mathrm{cat}(X_{i+1})$.*

Theorem 4.23 (I. [30])**.** *Let X be a space with $\mathrm{cat}(X_i) = m$, $i, m \geq 1$. We further assume that $\mathrm{Ext}(H_{i+1}(X), H_2(X){\otimes}H_{i+1}(X)) = 0$ or $m \geq 2$. Then $\mathrm{cat}(X_{i+1}) = \mathrm{cat}(X_i)$ implies $H_m(f_i) \ni 0$.*

Theorem 4.24 (I. [30]). *Let X is $d{-}1$ connected such that* $\operatorname{cat}(X_i) = m$, $\operatorname{cat}(X_{i+1}) = m{+}1$ *and* $\dim X_i \leq d(m+1) - 2$ *for some* $i \geq 1$. *Then* $\operatorname{cat}(X_{i+1}{\times}S^n) = \operatorname{cat}(X_{i+1})$ *implies* $\Sigma^n_* H_m(f_i) \ni 0$.

A few months after started to climb up the narrow unstable path, the author was next to Theorem 4.24 overlooking the necessity of many suspensions. If it were true without taking suspensions, we could get an affirmative answer for a wide range of spaces to the Ganea conjecture. However, once we have obtained the true statement, it then turned to convince us the existence of the counter-examples. The construction suggested by the statement is not very hard.

Theorem 4.25 (I. [28]). *There is a family of spaces* $\{Q_\ell\,;\,\ell \text{ is a prime} \geq 2\}$ *such that*

$$\begin{cases} \operatorname{cat}(Q_2{\times}S^n) = \operatorname{cat}(Q_2) & \text{for all } n \geq 1, \\ \operatorname{cat}(Q_\ell{\times}S^n) = \operatorname{cat}(Q_\ell) & \text{for all } n \geq 2 \text{ and } \ell > 2. \end{cases}$$

Outline of the proof of the case $\ell = 2$. Let $\sigma \in \pi_{15}(S^8)$ be the Hopf element. Then we have $H_1(\sigma) = 1 \in \pi_{15}(\Omega S^8{*}\Omega S^8) \cong \mathbb{Z}$. By Toda [74], the Whitehead product $[\iota_{15}, \iota_{15}] \in \pi_{29}(S^{15})$ is a non-trivial suspension element. Then Proposition 4.15 implies $H_1(\sigma\circ[\iota_{15}, \iota_{15}]) = i_*[\iota_{15}, \iota_{15}]$, where $i : S^{15} \hookrightarrow \Omega S^8{*}\Omega S^8$ is the bottom-cell inclusion. On the other hand, the bottom-cell inclusion induces a split monomorphism i_*, since $\Omega S^8{*}\Omega S^8$ has the homotopy type of a wedge of countably many spheres. Hence we have $H_1(\sigma\circ[\iota_{15}, \iota_{15}]) \neq 0$, and hence $Q_2 = S^8 \cup_{\sigma\circ[\iota_{15},\iota_{15}]} e^{30}$ satisfies $\operatorname{cat}(Q_2) = 2$ by Theorem 2.7.

On the other hand, we have $\Sigma([\iota_{15}, \iota_{15}]) = 0$, since a suspension of a Whitehead product is always 0. The cellular decomposition $Q_2{\times}S^n = Q_2{\times}\{*\} \cup S^8{\times}S^n \cup_{\psi_n} e^{n+30}$ implies that $\operatorname{cat}(Q_2{\times}\{*\} \cup S^8{\times}S^n) = 2$ and ψ_n is given by a relative Whitehead product of $\sigma\circ[\iota_{15}, \iota_{15}]$ and $\iota_n \in \pi_n(S^n)$. Then to get $\operatorname{cat}(Q_2{\times}S^n) = 3$, we need to show that $H_2(\psi_n) \not\ni 0$. However it is impossible, since $H_2(\psi_n) \ni S^{n-1}{*}H_1(\sigma\circ[\iota_{15}, \iota_{15}]) = \pm\Sigma^n[\iota_{15}, \iota_{15}] = 0$ for $n \geq 1$. Thus we have $\operatorname{cat}(Q_2{\times}S^n) = \operatorname{cat}(Q_2) = 2$ for all $n \geq 1$. □

This is not the end of this story, because L-S cat is defined on a smooth manifold, and there still is a possibility that the Ganea conjecture is true for a closed manifold. To study on this possibility, a sphere-bundle over a sphere must be the first test case, while the computation was left unknown as Ganea's Problem 4. Our observation using A_∞ method suggests that we need to compute the second or the third order higher Hopf invariant, which is closely related to the computation of Toda's secondary compositions - Toda brackets.

$\mathbb{C}\mathrm{P}^3 = S^2 \cup_\eta e^4 \cup e^6$ is one of the simplest non-trivial sphere bundles over a sphere - a S^2 bundle over S^4. For any map $\beta : S^q \to S^3$, we have a smooth approximation of $\Sigma\beta : S^{q+1} \to S^4$. Let $E(\beta)$ be the pull-back of $\mathbb{C}\mathrm{P}^3$ by the smooth map $\Sigma\beta : S^{q+1} \to S^4$. If β is non-trivial, we have $\operatorname{cat}(S^2 \cup_{\eta\circ\beta} e^{q+1}) = 2$, and we have a cellular decomposition of $E(\beta)$:

$$E(\beta) = S^2 \cup_{\eta\circ\beta} e^{q+1} \cup_{\psi(\beta)} e^{q+3}.$$

We can also show that H_2 of $\psi(\beta)$ is essentially given by $\Sigma^2\beta$. If $\Sigma^2\beta$ is non-trivial, we have $\mathrm{cat}(E) = 3 = \mathrm{cat}(E \cup (S^2 \cup_{\eta\circ\beta} e^{q+1}){\times}S^n)$ and we have a cellular decomposition of $E{\times}S^n$:

$$E{\times}S^n = E \cup (S^2 \cup_{\eta\circ\beta} e^{q+1}){\times}S^n \cup_{\Psi(\beta)} e^{n+q+3}.$$

So we are left to calculate H_3 of $\Psi(\beta)$. In fact, we can show that H_3 of $\Psi(\beta)$ is essentially given by $\Sigma^{n+2}\beta$, which can be performed using results by Toda [74] and Oka [55]:

Theorem 4.26 (I. [30, 31], L-S-V [49])**.** *There is a* 1 *connected closed manifold* N *such that*

$$\mathrm{cat}(N \smallsetminus \{*\}) = \mathrm{cat}(N).$$

Theorem 4.27 (I. [30, 31])**.** *There is a* 1 *connected closed manifold* M *such that*

$$\mathrm{cat}(M{\times}S^n) = \mathrm{cat}(M) \quad \textit{for all } n \geq 2.$$

These results shows that Problems 642 and 643 of [53] were solved in negative, even for a closed manifold. However all these examples still support the following conjecture.

Problem 4.28 (I. [28])**.** *Does* $n(X) = \mathrm{Max}\,\{n \mid \mathrm{cat}(X{\times}S^n) = \mathrm{cat}(X) + 1 \textit{ or } n = 0\}$ *satisfies the following equations?*

$$\mathrm{cat}(X{\times}S^n) = \begin{cases} \mathrm{cat}(X){+}1 & \textit{for all } n \leq n(X), \\ \mathrm{cat}(X) & \textit{for all } n > n(X). \end{cases}$$

After the publication of the Japanese version, the following result is published to give another supporting evidence of the above conjecture.

Theorem 4.29 (Stanley-Strom-I. [36])**.** *Let* X *be a* 1 *connected finite complex. If* $\mathrm{cat}(X{\times}S^{n'(X)}) = \mathrm{cat}(X)$ *for a sufficiently large* $n'(X) \gg 1$*, then we have*

$$\mathrm{cat}(X{\times}S^n) = \begin{cases} \mathrm{cat}(X){+}1 & \textit{for all } n \leq n(X), \\ \mathrm{cat}(X) & \textit{for all } n \geq n'(X), \end{cases}$$

while $n'(X)$ *is very large and depending on the connectivity, the dimension and the L-S cat of* X*.*

Also after the publication of the Japanese version, a result $\mathrm{cat}(\mathrm{Spin}(9)) = 8$ is announced by Kono-I. [45], which claims $\mathrm{cat}(\mathrm{Spin}(9)) = \mathrm{Mwgt}(\mathrm{Spin}(9); \mathbb{F}_2) = 8 > 6 = \mathrm{wgt}(\mathrm{Spin}(9); \mathbb{F}_2)$. This implies that $\mathrm{Mwgt}(\ ; \mathbb{F}_2)$ is actually a better invariant than $\mathrm{wgt}(\ ; \mathbb{F}_2)$.

Appendix A. L-S 'cat' of manifolds

(1) $\mathrm{cat}(\mathbb{F}P^n) = n,\ n{\geq}0$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$). (2) $\mathrm{cat}(\mathcal{O}P^n) = n,\ 0{\leq}n{\leq}2$.

(3) $\mathrm{cat}(M^2) = \begin{cases} 1, & \pi_1(M^2) = 0, \\ 2, & \pi_1(M^2) \neq 0. \end{cases}$ (4) $\mathrm{cat}(M^3) = \begin{cases} 1, & \pi_1(M^3) = 0, \\ 2, & \pi_1(M^3) = \text{a non-trivial free group}, \\ 3, & \text{otherwise}. \end{cases}$

(5) $\mathrm{cat}(S^n) = 1,\ n{\geq}1$. (6) (Krasnosel′skiĭ[46], GG [20]) $\mathrm{cat}(L^n(p)) = n,\ n{\geq}1, p{>}1$.

(7) (Rudyak [58, 59]) If a symplectic manifold (M^{2n}, ω) satisfies $\omega|_{\pi_2(M)}{=}0$, $\mathrm{cat}(M^{2n}) = 2n$.

(8) (Singhof [66], GG [20], I. [31]) S^r-bundle E over S^{t+1} and $Q = E{\smallsetminus}\{pt\} \simeq S^r{\cup_\alpha}e^{t+1}$.

r	t	α	cat$(Q{\times}S^n)$	cat(Q)	cat(E)	cat$(E{\times}S^n)$
$r{=}1$	$t{=}0$		2	1	2	3
	$t{=}1$	$\alpha{=}\pm 1$	1	0	1	2
		$\alpha{=}0$	2	1	2	3
		$\alpha{\neq}0, \pm 1$	3	2	3	4
	$t{>}1$		2	1	2	3
$r{>}1$	$t{<}r$		2	1	2	3
	$t{=}r$	$\alpha{=}\pm 1$	1	0	1	2
		$\alpha{\neq}\pm 1$	2	1	2	3
	$t{>}r$	$H_1(\alpha){=}0$	2	1	2	3
		$H_1(\alpha){\neq}0 \,\&\, \Sigma^r H_1(\alpha){=}0$	3 or 2	2	2	3
		$\Sigma^r H_1(\alpha){\neq}0$			3	3 or 4

(9) (Singhof [64, 65], JS [39], FGST [11], IM [33], IMN [34, 35]) Compact simple Lie groups.

rank	1		2		3		4		$5 \leq n$	
A_n	SU(2)	1	SU(3)	2	SU(4)	3	SU(5)	4	SU(n+1)	n
	PU(2)	3	PU(3)	6	PU(4)	9	PU(5)	12	PU(n+1)	$\leq 3n$
B_n	Spin(3)	1	Spin(5)	3	Spin(7)	5	Spin(9)	?	Spin($2n$+1)	?
	SO(3)	3	SO(5)	8	SO(7)	11	SO(9)	20	SO($2n$+1)	?
C_n	Sp(1)	1	Sp(2)	3	Sp(3)	5	Sp(4)	?	Sp(n)	?
	PSp(1)	3	PSp(2)	8	PSp(3)	?	PSp(4)	?	PSp(n)	?
D_n					Spin(6)	3	Spin(8)	6	Spin($2n$)	?
					SO(6)	9	SO(8)	12	SO($2n$)	?
					PSO(6)	9	PSO(8)	18	PSO($2n$)	?
Expt			$\mathrm{G_2}$	4			$\mathrm{F_4}$	?	$\mathrm{E_n}$	?

(10) (Singhof [64, 65]) Complex Stiefel manifold $W_{n,r} = \mathrm{U}(n)/\mathrm{U}(n{-}r)$ satisfies $\mathrm{cat}(W_{n,r}) = r$.

In addition, results on PU(n+1) is obtained by using the following recent result.

(11) (Kadzisa [41, 42]) $\mathrm{Cat}(\mathrm{U}(n)) = n$ $(= \mathrm{cat}(\mathrm{U}(n)) = \mathrm{cup}(\mathrm{U}(n)))$,

$\mathrm{Cat}(\mathrm{SU}(n)) = n{-}1$ $(= \mathrm{cat}(\mathrm{SU}(n)) = \mathrm{cup}(\mathrm{SU}(n)))$.

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Notes

1) In ordinary homotopy theory, e is used for Adams e-invariant, c for Chern class/number.

2) In this article, cup denotes a cup-length for some cohomology theory.

3) In this article, wgt denotes a category weight in the sense of Rudyak [58, 59] and Strom [71].

4) We can replace 'a closed subset' by 'a closed subset with homotopy extension property' for the same reason as the one given just after Definition 1.3

5) This well-known fact is obtained immediately from the definition of Cat by Ganea [15] or the proof of corresponding results of Varadarajan [77] and Hardie [21] for cat.

6) Ganea's original paper proves Theorem 2.3 and Theorem 2.2 is its corollary.

7) Please refer Section 3.

8) Please refer a table in Appendix.

9) While this result is stated in [37], its proof can be found in [35].

10) Please refer §4.1 for wgt$(-)$.

11) There apparently are two versions of A_m forms, [68] and [69]. The standard A_m structure does depend on the higher degeneracy conditions which is asserted only in [68]. The author does not know the two definitions are the actually the same or not.

12) We adopt this notation in this article to respect the originator, while we know that some people want to call this σcat. In [59], it is called r.

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(Norio Iwase, Fac. of Math., Kyushu Univ.)