A short proof for $\mathrm{tc}(K) = 4$

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Abstract

We show a method to determine topological complexity from the fibrewise view point, which provides an alternative proof for $\mathrm{tc}(K) = 4$, where K denotes Klein bottle.

Keywords: topological complexity, fibrewise homotopy theory, A_∞ structure, Lusternik-Schnirelmann category, module weight.

1. Introduction

The topological complexity is introduced in [Far03] by M. Farber for a space X and is denoted by $\mathrm{TC}(X)$: $\mathrm{TC}(X)$ is the minimal number $m \geq 1$ such that $X \times X$ is covered by m open subsets U_i $(1 \leq i \leq m)$, each of which admits a continuous section $s_i : U_i \to \mathcal{P}(X) = \{u : [0,1] \to X\}$ for the fibration $\varpi : \mathcal{P}(X) \to X \times X$ given by $u \mapsto (u(0), u(1))$. Similarly, the monoidal topological complexity of X denoted by $\mathrm{TC}^{\mathrm{M}}(X)$ is the minimal number $m \geq 1$ such that $X \times X$ is covered by m open subsets $U_i \supset \Delta X$ $(1 \leq i \leq m)$, each of which admits a section $s_i : U_i \to \mathcal{P}(X)$ of $\varpi : \mathcal{P}(X) \to X \times X$ such that $s_i(x,x)$ is the constant path at x for any $(x,x) \in U_i \cap \Delta X$. In this paper, we denote $\mathrm{tc}(X) = \mathrm{TC}(X) - 1$ and $\mathrm{tc}^{\mathrm{M}}(X) = \mathrm{TC}^{\mathrm{M}}(X) - 1$.

Let $E = (E, B; p, s)$ be a fibrewise pointed space, i.e, $p : E \to B$ is a fibrewise space with a section $s : B \to E$. For a fibrewise pointed space $E' = (E', B'; p', s')$ and a fibrewise pointed map $f : E' \to E$, we have pointed and unpointed versions of fibrewise L-S category, denoted by $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f)$ and $\mathrm{cat}^{*}_{\mathrm{B}}(f)$, respectively: $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f)$ is the minimal number $m \geq 0$ such that E' is covered by $(m+1)$ open subsets U_i and $f_i = f|_{U_i}$ is fibrewise pointedly fibrewise compressible into $s(B)$, and $\mathrm{cat}^{*}_{\mathrm{B}}(f)$ is the minimal number $m \geq 0$

such that E is covered by $(m{+}1)$ open subsets U_i and $f_i = f|_{U_i}$ is fibrewise-unpointedly fibrewise compressible into $s(B)$. We denote $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(\mathrm{id}_E) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(E)$ and $\mathrm{cat}^{*}_{\mathrm{B}}(\mathrm{id}_E) = \mathrm{cat}^{*}_{\mathrm{B}}(E)$ (see [IS10]). Then by definition, $\mathrm{tc}(X) \leq \mathrm{tc}^{\mathrm{M}}(X)$ for a space X, $\mathrm{cat}^{*}_{\mathrm{B}}(E) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(E)$ for a fibrewise pointed space E, and $\mathrm{cat}^{*}_{\mathrm{B}}(f) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f)$, $\mathrm{cat}^{*}_{\mathrm{B}}(f) \leq \mathrm{cat}^{*}_{\mathrm{B}}(E)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(f) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(E)$ for a fibrewise pointed map $f : E' \to E$.

In [Sak10], the m-th fibrewise projective space $P^m_B\Omega_B E$ of a fibrewise loop space $\Omega_B E$ is introduced and characterized with a natural map $e^E_m : P^m_B\Omega_B E \to E$. Using them, we can characterise numerical invariants in [IS10]: firstly, the fibrewise cup-length $\mathrm{cup}_{\mathrm{B}}(E;h)$ is given by

$$\max\left\{m{\geq}0 \,\middle|\, \exists_{\{u_1,\cdots,u_m\}\subset H^*(E,s(B))}\ u_1\cdots u_m \neq 0\right\}.$$

Secondly, the fibrewise categorical weight $\mathrm{wgt}_{\mathrm{B}}(E;h)$ is the smallest number m such that $e^E_m : P^m_B\Omega_B E \to E$ induces a monomorphism of generalised cohomology theory h^*. Thirdly, the fibrewise module weight $\mathrm{Mwgt}_{\mathrm{B}}(E;h)$ is the least number m such that $e^E_m : P^m_B\Omega_B E \to E$ induces a split monomorphism of generalised cohomology theory h^* as an h_*h-module. The latter two invariants are versions of categorical weight introduced by Rudyak [Rud98] and Strom [Str00] whose origin is in Fadell-Husseini [FH92].

Theorem 1.1. *$\mathrm{cup}_{\mathrm{B}}(E;h) \leq \mathrm{wgt}_{\mathrm{B}}(E;h) \leq \mathrm{Mwgt}_{\mathrm{B}}(E;h) \leq \mathrm{cat}^{*}_{\mathrm{B}}(E) \leq \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(E)$.*

Proof. Let $\mathrm{cat}^{*}_{\mathrm{B}}(E) = m$. Then there is a covering of E with $m{+}1$ open subsets $\{U_i | 0 \leq i \leq m\}$ such that each U_i can be compressed into $s(B) \subset E$. So, there is an unpointed fibrewise homotopy of $\mathrm{id} : E \to E$ to a map $r_i : E \to E$ satisfying $r_i(U_i) \subset s(B)$, which gives an unpointed fibrewise compression of the fibrewise diagonal $\Delta_B : E \to \prod^{m+1}_B E$ into the fibrewise fat wedge $\prod^{[m+1]}_B E \subset \prod^{m+1}_B E$. Since a continuous construction on a space can be extended on a cell-wise trivial fibrewise space by [IS08], the fibrewise projective m-space $P^m_B\Omega_B E$ has the fibrewise homotopy type of the fibrewise homotopy pull-back of $\Delta_B : E \to \prod^{m+1}_B E$ and the inclusion $\prod^{[m+1]}_B E \subset \prod^{m+1}_B E$. Hence by James-Morris [JM91], we have a map $\sigma : E \to P^m_B\Omega_B E$ which is an unpointed fibrewise homotopy inverse of $e^E_m : P^m_B\Omega_B E \to E$, and hence we obtain $\mathrm{Mwgt}_{\mathrm{B}}(E;h) \leq m = \mathrm{cat}^{*}_{\mathrm{B}}(E)$. Combining this with [IS10, Theorem 8.6][1], we obtain the theorem. □

[1]As is mentioned in [IS12], the equality of tc^{M} and tc stated in [IS10, Theorem 1.13]

From now on, we assume that $(E, B; p, s)$ is given by $E = X \times X$, $B = X$, $p = \mathrm{proj}_1 : X \times X \to X$ and $s = \Delta : X \to X \times X$ the diagonal map, and so we have $\mathrm{cat}^*_{\mathrm{B}}(E) = \mathrm{tc}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(E) = \mathrm{tc}^{\mathrm{M}}(X)$ by [IS10, IS12]. Hence we obtain the following by Theorem 1.1.

Theorem 1.2. $\mathrm{wgt}_{\mathrm{B}}(E; h) \leq \mathrm{Mwgt}_{\mathrm{B}}(E; h) \leq \mathrm{tc}(X) \leq \mathrm{tc}^{\mathrm{M}}(X)$.

If h is the ordinary cohomology with coefficients in R, we write $\mathrm{cup}_{\mathrm{B}}(E; h)$, $\mathrm{wgt}_{\mathrm{B}}(E; h)$ and $\mathrm{Mwgt}(E; h)$ as $\mathrm{cup}_{\mathrm{B}}(E; R)$, $\mathrm{wgt}_{\mathrm{B}}(E; R)$ and $\mathrm{Mwgt}(E; R)$, respectively. We might disregard R later in this paper, if $R = \mathbb{F}_2$ the prime field of characteristic 2.

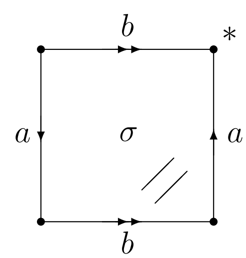
As an application, we give an alternative proof of a result recently announced by several authors. Let K_q be the non-orientable closed surface of genus $q \geq 1$, and denote $K = K_2$.

Theorem 1.3 (Cohen-Vandembroucq [CV])**.** *For $q \geq 2$, we have* $\mathrm{wgt}(K_q) = \mathrm{Mwgt}(K_q) = \mathrm{tc}(K_q) = \mathrm{tc}^{\mathrm{M}}(K_q) = 4$ *and* $\mathrm{TC}(K_q) = \mathrm{TC}^{\mathrm{M}}(K_q) = 5$.

Corollary 1.4. *The fibration $S^1 \hookrightarrow K \to S^1$ is an example answering a question by negative, which is raised by Mark Grant in [Gra12]: is* $\mathrm{TC}(E) \leq \mathrm{TC}(F) \times \mathrm{TC}(B)$ *always true for a fibration $F \to E \to B$?*

2. Fibrewise Resolution of Klein Bottle

For $q \geq 1$, $\pi_1(K_q)$ is given by $\pi_1^q = \langle b, b_1, \ldots, b_{q-1} \,|\, b_1^2 \cdots b_{q-1}^2 = b^2 \rangle$. We know that K_q is a CW complex with one 0-cell $*$, q 1-cells $b, b_1, \ldots, b_{q-1}$ and one 2-cell σ_q.



For $a = b_1 b^{-1}$, we know $\pi_1^2 = \{a^k b^\ell \,|\, k, \ell \in \mathbb{Z}\}$ with a relation $aba = b$. Let us denote $\varepsilon(\ell) = \frac{1-(-1)^\ell}{2}$, which is either 0 or 1, to obtain $a^{k_1} b^{\ell_1} a^{k_2} b^{\ell_2} = a^{k_1 + k_2 - 2\varepsilon(\ell_1) k_2} b^{\ell_1 + \ell_2}$, $b^{\pm 1}(a^k b^\ell) b^{\mp 1} = a^{-k} b^\ell$ and $a^{\pm 1}(a^k b^\ell) a^{\mp 1} = a^{k \pm 2\varepsilon(\ell)} b^\ell$. We denote $\bar{\tau} = \tau^{-1}$ to simplify expressions. We know the multiplication of $\pi_1^2 = \pi_0(\Omega K_2)$ is inherited from the loop addition. Hence the natural equivalence $\Omega K_2 \to \pi_1^2$ is an A_∞-map, since a discrete group has no non-trivial higher structure on a given multiplication.

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is appeared to be an open statement. But the inequality in [IS10, Theorem 8.6] does not depend on the open statement.

Let $E_q = (E_q, B_q; p_q, s_q)$ be the fibrewise pointed space, where $E_q = K_q{\times}K_q$, $B_q = K_q$, $p_q = \mathrm{proj}_1 : K_q{\times}K_q \to K_q$ and $s_q = \Delta : K_q \to K_q{\times}K_q$. When $q{=}2$, we abbreviate E_2, K_2, σ_2 and π^2_1 as E, K, σ and π, respectively in this paper.

Let $\widetilde{K} = \bigcup_{\boldsymbol{a}\in K} \pi_1(K;\boldsymbol{a},*) \to K$ be the universal covering of K, and $\widehat{K} = \widetilde{K}{\times}_{\mathrm{ad}}\pi \to K$ be the associated covering space, where 'ad' is the equivalence relation on $\widetilde{K}{\times}\pi$ given by $([\kappa{\cdot}\lambda], g) \sim ([\kappa], hgh^{-1})$ for g, $h = [\lambda] \in \pi$ and $[\kappa] \in \pi_1(K;\boldsymbol{a},*)$. We regard $\widehat{K} = \bigcup_{\boldsymbol{a}\in K} \pi_1(K,\boldsymbol{a})$. Since the fibrewise pointed space $\widehat{K}{=}\widetilde{K}{\times}_{\mathrm{ad}}\pi \to K$ is a fibrewise discrete group over K, it has a fibrewise projective space by [Sak10]. In this paper, we define $P^m_B\widehat{K} = \widetilde{K}{\times}_{\mathrm{ad}}P^m\pi$ as a fibrewise projective space, where the adjoint action is given as follows:

$$h[g_1|g_2|\cdots|g_m] = [hg_1h^{-1}|hg_mh^{-1}]$$

for $h \in \pi,\ [g_1|g_2|\cdots|g_m] \in P^m\pi$. By the definition given above, $P^\infty_B\widehat{K}$ might be considered as the fibrewise Bar construction of $\widehat{K}$ over K, since the fibre $P^\infty\pi = B\pi$ is the Bar construction of π, where π is the fibre of $\widehat{K}$ over K.

Proposition 2.1 (Example 6.2 (4) of [IS10]). *$P^m_B\Omega_BE \simeq_B P^m_B\widehat{K}$ for all $m{\geq}1$.*

Proof. For $[\gamma]{=}g{\in}\pi$, we denote by Ω^g_BE and $\widehat{K}^g$ the connected components of $\gamma \in \Omega_BE$ and $([*], g) \in \widetilde{K}{\times}_{\mathrm{ad}}\pi = \widehat{K}$, respectively. Then the image of $\pi_1(\Omega^g_BE)$ in $\pi_1(K)$ is the centralizer of g, which is the same as $\pi_1(\widehat{K}^g)$. Thus, there is a lift $\widehat{\Omega^g_Bp} : \Omega^g_BE \to \widehat{K}^g$ of $\Omega^g_Bp = \Omega_Bp|_{\Omega^g_BE} : \Omega^g_BE \to K$ whose restriction to the fibre on $\boldsymbol{a}$ is the natural map : $\Omega(K,\boldsymbol{a}){\cap}\Omega^g_BE{\to}\pi_1(K,\boldsymbol{a}){\cap}\widehat{K}^g$. Hence we obtain a lift $\widehat{\Omega_Bp} : \Omega_BE \to \widehat{K}$ of $\Omega_Bp : \Omega_BE \to K$ given by $\widehat{\Omega_Bp}|_{\Omega^g_BE} = \widehat{\Omega^g_Bp}$, whose restriction to the fibre on $\boldsymbol{a}$ is the natural map : $\Omega(K,\boldsymbol{a}) \to \pi_1(K,\boldsymbol{a})$. Moreover, the restriction of $\widehat{\Omega_Bp}$ to each fibre is a pointed homotopy equivalence since K is a $K(\pi,1)$ space. Then by Dold [Dol55], $\widehat{\Omega_Bp} : \Omega_BE \to \widehat{K}$ is a fibrewise homotopy equivalence. Here, since the section : $K \to \Omega^e_BE$ of $\Omega^e_Bp : \Omega^e_BE \to K$ given by trivial loops is a fibrewise cofibration, $\widehat{\Omega_Bp}$ is a fibrewise pointed homotopy equivalence by James [Jam95]. Moreover, $\widehat{\Omega_Bp}$ is a fibrewise A_∞-map since each fibre of $\widehat{K}{\to}K$ is a discrete set. Thus $P^m_B\Omega_BE \simeq_B P^m_B\widehat{K}$, $m{\geq}1$. □

Now, we are ready to give the cell decomposition of $P^m_B\widehat{K} \simeq_B P^m_B\Omega_BE$.

Firstly, the cell structure of K is given as follows: let $\Lambda_0 = \{*\}$, $\Lambda_1 = \{a, b\}$, $\Lambda_2 = \{\sigma\}$.

$$K = \bigcup_{0\le k\le 2}\bigcup_{\eta\in\Lambda_k} e^k_\eta = e^0_* \cup e^1_a \cup e^1_b \cup e^2_\sigma.$$

From now on, e^k_η will be denoted by $[\eta]$ for $\eta \in \Lambda_k$, which is in the cellular chain group $\mathbb{Z}\Lambda = \mathbb{Z}\{*, a, b, \sigma\}$, $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$. The boundary of $[\eta]$ for $\eta \in \Lambda_k$ is expressed in $\mathbb{Z}\Lambda$ as follows:

$$\partial[\eta] = [\partial\eta], \quad \partial * = 0, \quad \partial a = 0, \quad \partial b = 0 \ \text{ and } \ \partial\sigma = 2a,$$

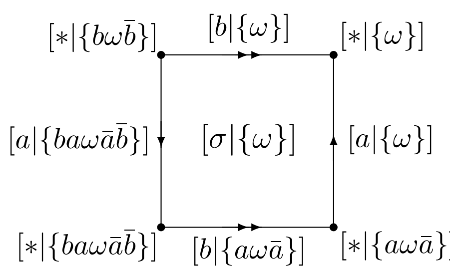
Secondly, $P^m\pi$ is a Δ-complex in the sense of Hatcher [Hat02]:

$$P^m\pi = \bigcup_{0\le n\le m}\ \bigcup_{\omega=(g_1,\ldots,g_n)\in\pi^n} e^n_\omega,$$

In this paper, e^n_ω will be denoted by $[\omega]$ or $[g_1|\cdots|g_n]$ for $\omega = (g_1, \ldots, g_n)$ which is in the cellular chain group $\overset{m}{\underset{n=0}{\oplus}} \otimes^n \mathbb{Z}\pi \cong \overset{m}{\underset{n=0}{\oplus}} \mathbb{Z}\pi^n$. The boundary of $[\omega]$ is expressed as follows:

$$\partial[\omega] = [\partial\omega], \qquad \partial_i\omega = \begin{cases} \partial_0\omega = (g_2, \ldots, g_n), & i=0, \\ (g_1, \ldots, g_i g_{i+1}, \cdots, g_n), & 0<i<n, \\ \partial_n\omega = (g_1, \ldots, g_{n-1}), & i=n, \end{cases}$$
$$\partial\omega = \sum_{i=0}^{n}(-1)^i\partial_i\omega,$$

which coincides with the boundary in m-th filtration of Bar resolution of π.



For $\tau \in \Lambda_1$, and $\omega \in \pi^n$, $[\bar\tau|\{\omega\}]$ represents the same product cell as $[\tau|\{\bar\tau\omega\tau\}]$ with orientation reversed, and we have $[\bar\tau|\{\omega\}] = -[\tau|\{\bar\tau\omega\tau\}]$, $\bar\tau(g_1, \ldots, g_n)\tau = (\bar\tau g_1\tau, \ldots, \bar\tau g_n\tau)$. To observe this, let us look at the end point of τ, where the fibre lies: A 1-cell τ is a path $\tau : I = [0, 1] \to K$ which has a lift to a path $\tilde\tau : I \to \widehat{K}$ with an initial data $[\lambda] \in \pi_1(K, \tau(1))$ given by $\tilde\tau(t) = [\tau_t{\cdot}\lambda{\cdot}\tau_t^{-1}] \in \pi_1(K, \tau(t))$, where we denote $\tau_t(s) = \tau(t+(1-t)s)$.

Thirdly, since $\Omega_B E$ is fibrewise A_∞-equivalent to $\widehat{K}$, $P^m_B\Omega_B E$ is fibrewise pointed homotopy equivalent to $P^m_B\hat{K}$. A $k{+}n$-cell of $P^m_B\Omega_B E \simeq_B P^m_B\widehat{K} = \widetilde{K}\times_{\mathrm{ad}}P^m\pi$ is described as a product cell of a k-cell $[\eta]$ in K and a Δ n-cell $[\omega]$ in $P^m\pi$, and is denoted by $e^{n+k}_{(\eta;\omega)} \approx \mathrm{Int}(\Box^k)\times \mathrm{Int}(\Delta^n)$.

$$P^m_B\Omega_B E \simeq_B P^m_B\widehat{K} = \bigcup_{0\le n\le m}\ \bigcup_{\omega\in\pi^n} \left(e^n_{(*;\omega)} \cup e^{n+1}_{(b;\omega)} \cup e^{n+1}_{(b_1;\omega)} \cup e^{n+2}_{(\sigma;\omega)}\right).$$

In this paper, $e^{n+k}_{(\eta;\omega)}$ will be denoted by $[\eta|\{\omega\}]$ or $[\eta|\{g_1|\cdots|g_n\}]$, for $(\eta;\omega)=(\eta;g_1,\ldots,g_n)\in\Lambda_k\times\pi^n$, in the cellular chain group $C^*(P^m_B\widehat{K};\mathbb{Z}) = \bigoplus_{n=0}^{m}\mathbb{Z}\,\Lambda_0\times\pi^n \oplus \bigoplus_{n=1}^{m+1}\mathbb{Z}\,\Lambda_1\times\pi^{n-1} \oplus \bigoplus_{n=2}^{m+2}\mathbb{Z}\,\Lambda_2\times\pi^{n-2}$.

Let $[\omega]=[g_1|g_2|\cdots|g_n]$ be a Δ n-cell in $P^m\widehat{K}$ with $g_i\in\pi_1(K,\tau(1))$. Then the boundary of a product cell $[\tau|\{\omega\}]$ of ω with a 1-cell $[\tau]$ of K is the union of cells $[\tau|\{\partial_i\omega\}]$, $0\le i\le n$, $[\omega]$ and $[\tau\omega\bar{\tau}]=[\tau g_1\bar{\tau}|\tau g_2\bar{\tau}|\cdots|\tau g_n\bar{\tau}]$. Similarly, the boundary of a product cell $[\sigma|\{\omega\}]$ of ω with a 2-cell $[\sigma]$ of K is the union of cells $[\sigma|\{\partial_i\omega\}]$, $0\le i\le n$, $[a|\{\omega\}]$, $[b|\{\omega\}]$, $[a|\{ba\omega\bar{a}\bar{b}\}]$ and $[b|\{a\omega\bar{a}\}]$.

Then the modulo 2 boundary formula of a cell in $P^m_B\widehat{K}$ in the cellular chain group $C^*(P^m_B\widehat{K};\mathbb{Z}/2\mathbb{Z})$ is given by the following, where, for any $m,n\in\mathbb{Z}$ and $p\ge 2$, $m\underset{(p)}{=}n$ implies that m is equal to n modulo p.

Proposition 2.2. 1. *Since* $\partial[\tau|\{\omega\}]=[*|\{\omega\}]\cup[*|\{\tau\omega\bar{\tau}\}]\cup\bigcup_{0\le i\le n}[\tau|\{\partial_i\omega\}]$, *we have* $\partial[\tau|\{\omega\}] \underset{(2)}{=} [*|\{\omega\}]+[*|\{\tau\omega\bar{\tau}\}]+[\tau|\{\partial\omega\}]$ *for* $\tau\in\Lambda_1$ *and* $\omega\in\pi^n$, *where* $[\tau|\{\partial\omega\}]=\sum_{i=0}^{n}(-1)^i[\tau|\{\partial_i\omega\}]$.

2. *Since* $\partial[\sigma|\{\omega\}]=[a|\{\omega\}]\cup[a|\{ba\omega\bar{a}\bar{b}\}]\cup[b|\{\omega\}]\cup[b|\{a\omega\bar{a}\}]\cup\bigcup_{0\le i\le n}[\sigma|\{\partial_i\omega\}]$, *we have* $\partial[\sigma|\{\omega\}] \underset{(2)}{=} [a|\{\omega\}]+[a|\{ba\omega\bar{a}\bar{b}\}]+[b|\{\omega\}]+[b|\{a\omega\bar{a}\}]+[\sigma|\{\partial\omega\}]$ *for* $\omega\in\pi^n$, *where* $[\sigma|\{\partial\omega\}] = \sum_{i=0}^{n}(-1)^i[\sigma|\{\partial_i\omega\}]$.

3. Topological Complexity of non-orientable surface

Since $P^\infty\pi\simeq K$, we have $H^*(P^\infty\pi)=\mathbb{F}_2\{1,x,y,z\}$ with $z=xy=yx=x^2$, where x, y are dual to $[a]$, $[b]$, respectively, the generators of $H_1(P^\infty\pi)\cong\mathbb{F}_2[a]\oplus\mathbb{F}_2[b]$. We regard x and y are in $Z^1(P^\infty\pi)$ and $z=x\cup y$ is in $Z^2(P^\infty\pi)$. A simple computation shows that $[a^kb^\ell]$ is homologous to $k[a]+\ell[b]$ in $Z_1(P^\infty\pi)$, and we have $x[a^kb^\ell]=k$ and $y[a^kb^\ell]=\ell$. By definition of a cup product in a chain complex, we obtain the following equality:

$$z[a^{k_1}b^{\ell_1}|a^{k_2}b^{\ell_2}]=(x\cup y)[a^{k_1}b^{\ell_1}|a^{k_2}b^{\ell_2}]=x[a^{k_1}b^{\ell_1}]\cdot y[a^{k_2}b^{\ell_2}]=k_1\ell_2 \quad \text{in } P^m\pi,$$

where we denote $x|_{P^m\pi}$, $y|_{P^m\pi}$ and $z|_{P^m\pi}$ again by x, y and z, respectively.

Proposition 3.1. 1. $e^K_m : P^m\pi\hookrightarrow P^\infty\pi \overset{\simeq}{\to} K$ *induces, up to dimension 2 in the ordinary* $\mathbb{F}_2$*-cohomology, a monomorphism if* $m\ge 2$*, and an isomorphism if* $m\ge 3$.

2. $e^E_m : P^m_B\widehat{K} \hookrightarrow P^\infty_B\widehat{K} \overset{\simeq}{\to} E$ *induces, up to dimension* 4 *in the ordinary* $\mathbb{F}_2$*-cohomology, a monomorphism if* $m \geq 4$*, and an isomorphism if* $m \geq 5$.

Proof. Since $P^m\pi$ is the m-skeleton of $P^\infty\pi$, the pair $(P^\infty\pi, P^m\pi)$ is m-connected, and so is the fibrewise pair $(P^\infty_B\widehat{K}, P^m_B\widehat{K})$ over K. It implies the proposition. □

By Proposition 3.1 (1), we can easily see the following propostion.

Proposition 3.2. *The cocycle* z *represents the generator of* $H^2(P^m\pi) \cong \mathbb{F}_2$ *for* $m \geq 3$.

Associated with the filtration $\{F_i(m) = p_m^{-1}(K^{(i)})\}$ of $P^m_B\widehat{K} \simeq_B P^m_B\Omega_B E$, given by the CW filtration $\{*\} = K^{(0)} \subset K^{(1)} \subset K^{(2)} = K$ of K with $K^{(1)} = \{*\} \cup e^1_{(a)} \cup e^1_{(b)} \approx S^1 \vee S^1$, we have Serre spectral sequence $E^{*,*}_r(m) = E^{*,*}_r(P^m_B\widehat{K})$ converging to $H^*(P^m_B\widehat{K})$ with $E^{p,q}_1(m) \cong H^{p+q}(F_p(m), F_{p-1}(m)) \cong H^p(K^{(p)}, K^{(p-1)}; H^q(P^m\pi))$ the cohomology with local coefficients.

From now on, we denote $\alpha = (a^{k_1}b^{\ell_1})$, $\tau = (a^{k_1}b^{\ell_1}, a^{k_2}b^{\ell_2})$ and $\omega = (a^{k_1}b^{\ell_1}, a^{k_2}b^{\ell_2}, a^{k_3}b^{\ell_3})$. Let functions : $[a^{k_1}b^{\ell_1}|\cdots|a^{k_n}b^{\ell_n}] \mapsto k_i$ and ℓ_i by (k_i) and (ℓ_i), respectively for $1 \leq i \leq n$. Then for a function $f : \mathbb{Z}^{2n} \to \mathbb{Z}$, we obtain a function $(f(\{k_i\}, \{\ell_i\})) : [a^{k_1}b^{\ell_1}|\cdots|a^{k_n}b^{\ell_n}] \mapsto f(\{k_i\}, \{\ell_i\})$. By Proposition 3.1, $H^4(P^5_B\widehat{K}) \cong \mathbb{F}_2$ is generated by $(e^E_5)^*([z \otimes z])$, which comes from $E^{2,2}_1(5) \cong H^4(F_2(5), F_1(5))$ for dimensional reasons. By the isomorphism $H^4(F_2(5), F_1(5)) \cong H^2(P^5\pi)$, $(e^E_5)^*[z \otimes z]$ corresponds to $[z] \in H^2(P^5\pi)$ by Proposition 3.2, and hence a representing cocycle $w \in Z^4(P^5_B\widehat{K})$ of $(e^E_5)^*[z \otimes z]$ can be chosen as a homomorphism defined by the formulae

$$w[\sigma|\{\tau\}] = z[\tau] = k_1\ell_2, \quad w|_{F_1(5)} = 0.$$

When $3 \leq m \leq 5$, we denote $w|_{P^m_B\widehat{K}}$ again by $w \in Z^4(F_2(m), F_1(m))$, which is representing a generator of $E^{2,2}_1(m)$. Furthermore, $[w] \neq 0$ in $E^{2,2}_\infty(m)$ if $m \geq 4$ by Proposition 3.1.

Our main goal is to show $[w] = 0$ in $H^*(P^3_B\widehat{K})$: we remark here that $\varepsilon(\ell) \underset{(2)}{=} \ell$ for $\ell \in \mathbb{Z}$, since $\varepsilon(\ell) = 0 \iff (-1)^\ell = 1 \iff \ell$ is even.

Firstly, let us introduce a numerical function given by the floor function.

Defnінition 3.3. $t(m) = \lfloor \frac{m}{2} \rfloor$ for $m \in \mathbb{Z}$.

Then we have $t(0) = 0$ and we obtain the following.

Proposition 3.4. 1. $t(-m) \underset{(2)}{=} t(m) + m$,

2. $t(m+n+2\ell) \underset{(2)}{=} t(m)+t(n)+mn+\ell,$ *for* $m,n,\ell\in\mathbb{Z}$.

Proof. This proposition can be obtained by straight-forward calculations, and so we left it to the reader. $\square$

Corollary 3.5. 1. $t(k_1)[\partial\tau]=t(k_2)+t(k_1+k_2+2\varepsilon(\ell_1)k_2)+t(k_1)=(\ell_1+k_1)k_2,$

2. $(k_1t(k_2))[\partial\omega]=k_2t(k_3)+(k_1+k_2)t(k_3)+k_1t(k_2+k_3+2\varepsilon(\ell_2)k_3)+k_1t(k_2)= k_1(\ell_2+k_2)k_3.$

Secondly, let an element $u\in C^3(P^3_B\widehat{K})$ be given by the formulae below:

$$u[*|\{\omega\}] = k_1t(k_2)\ell_3k_3 + k_1(\ell_2k_3+k_2\ell_3+k_2)t(k_3),$$

$$u[a|\{\tau\}] = 0,\quad u[b|\{\tau\}] = (k_1t(k_2))[\tau] \text{ and } u[\sigma|\{\alpha\}] = 0.$$

Then δu enjoys the following formulae by Propositions 2.2, 3.4 and Corollary 3.5 in $C^*(P^3_B\widehat{K})$:

1. $$\begin{aligned}(\delta u)[\sigma|\{\tau\}] &\underset{(2)}{=} u[a|\{\tau\}] + u[a|\{a^{-k_1-2\varepsilon(\ell_1)}b^{\ell_1}|a^{-k_2-2\varepsilon(\ell_2)}b^{\ell_2}\}]\\ &\quad + u[b|\{\tau\}] + u[b|\{a^{k_1+2\varepsilon(\ell_1)}b^{\ell_1}|a^{k_2+2\varepsilon(\ell_2)}b^{\ell_2}\}] + u[\sigma|\{\partial\tau\}]\\ &\underset{(2)}{=} 0 + k_1(t(k_2)+t(k_2+2\varepsilon(\ell_2))) + 0 \underset{(2)}{=} k_1\varepsilon(\ell_2) \underset{(2)}{=} k_1\ell_2 = w[\sigma|\{\tau\}].\end{aligned}$$

2. $$\begin{aligned}(\delta u)[a|\{\omega\}] &\underset{(2)}{=} u[*|\{\omega\}] + u[*|\{a^{k_1+2\varepsilon(\ell_1)}b^{\ell_1}|a^{k_2+2\varepsilon(\ell_2)}b^{\ell_2}|a^{k_3+2\varepsilon(\ell_3)}b^{\ell_3}\}]\\ &\quad +u[a|\{\partial\omega\}]\\ &\underset{(2)}{=} k_1(t(k_2)+t(k_2+2\varepsilon(\ell_2)))\ell_3k_3\\ &\quad +k_1(\ell_2k_3+k_2\ell_3+k_2)(t(k_3)+t(k_3+2\varepsilon(\ell_3))) + 0\\ &\underset{(2)}{=} k_1\varepsilon(\ell_2)\ell_3k_3 + k_1(\ell_2k_3+k_2\ell_3+k_2)\varepsilon(\ell_3) \underset{(2)}{=} 0 = w[a|\{\omega\}].\end{aligned}$$

3. $$\begin{aligned}(\delta u)[b|\{\omega\}] &\underset{(2)}{=} u[*|\{\omega\}] + u[*|\{a^{-k_1}b^{\ell_1}|a^{-k_2}b^{\ell_2}|a^{-k_3}b^{\ell_3}\}] + u[b|\{\partial\omega\}]\\ &\underset{(2)}{=} k_1(t(k_2)+t(-k_2))\ell_3k_3 + k_1(\ell_2k_3+k_2\ell_3+k_2)(t(k_3)+t(-k_3))\\ &\quad +(k_1t(k_2))[\partial\omega]\\ &\underset{(2)}{=} k_1k_2\ell_3k_3 + k_1(\ell_2k_3+k_2\ell_3+k_2)k_3 + k_1(\ell_2+k_2)k_3 \underset{(2)}{=} 0 = w[b|\{\omega\}].\end{aligned}$$

Thus we obtain that $\delta u \underset{(2)}{=} w$ in $C^*(P^3_B\widehat{K})$, which enables us to show the following.

Theorem 3.6. $\mathrm{tc}^{\mathrm{M}}(K) = \mathrm{tc}(K) = \mathrm{wgt}_{\mathrm{B}}(E) = \mathrm{wgt}_{\mathrm{B}}(z\otimes z) = 4.$

Proof. By the above arguments, we have $(e_3^E)^*(z\otimes z) = [w] = [\delta u] = 0$ in $H^*(P_B^3\widehat{K})$, and hence $0 \neq z\otimes z \in \ker(e_3^E)^*$ which implies $\mathrm{wgt}_{\mathrm{B}}(E) \geq \mathrm{wgt}_{\mathrm{B}}(z\otimes z) \geq 4$. On the other hand, Theorem 1.2 implies $\mathrm{wgt}_{\mathrm{B}}(E) \leq \mathrm{tc}(K) \leq \mathrm{tc}^{\mathrm{M}}(K) \leq 2\,\mathrm{cat}(K) \leq 2\dim K = 4$. It implies the theorem. □

Remark 3.7. Let $u_0 \in C^2(P_B^2\widehat{K})$ and $w_0 \in C^3(P_B^2\widehat{K})$ be as follows:

$$u_0[*|\{\tau\}] = (t(k_1)\ell_2k_2 + (\ell_1k_2+k_1\ell_2+k_1)t(k_2))[\tau], \quad u_0[a|\{\alpha\}] = 0,$$
$$u_0[b|\{\alpha\}] = t(k_1)[\alpha],\ u_0[\sigma|\{*\}] = 0; \quad w_0[\sigma|\{\alpha\}] = y[\alpha] = \ell_1,\ w_0|_{F_1(2)} = 0.$$

Then we can observe $\delta(u_0) \underset{(2)}{=} w_0$ and $[w_0]=0$ in $H^*(P_B^2\widehat{K})$, which would imply $\mathrm{wgt}_{\mathrm{B}}(z\otimes y) = 3$.

Let $q \geq 2$. Then by sending b to b, b_1 to ab, and all other b_i's to 1, $1<i<q$, we obtain a homomorphism $\phi_q : \pi_1^q \to \pi$, since $(ab)^2 = b^2$ in π. Then ϕ_q induces maps $B\phi_q : K_q = B\pi_q \to B\pi = K$ and $P_B^m\widehat{\phi_q} : P_B^m\widehat{K_q} \to P_B^m\widehat{K}$ such that $e_m^{E_q}\circ P_B^m\widehat{\phi_q} = (\phi_q\times\phi_q)\circ e_3^E$. Since $\phi_q^* : H^2(K) \to H^2(K_q)$ is an isomorphism, $z_q := \phi_q^*(z)$ is the generator of $H^2(K_q) \cong \mathbb{F}_2$. Hence $(e_3^{E_q})^*(z_q\otimes z_q) = (e_3^{E_q})^*\circ(\phi_q\times\phi_q)^*(z\otimes z) = (P_B^3\widehat{\phi_q})^*\circ(e_3^E)^*(z\otimes z) = 0$ by Theorem 3.6, and we obtain $4 \leq \mathrm{wgt}_{\mathrm{B}}(z_q\otimes z_q) \leq \mathrm{wgt}_{\mathrm{B}}(E_q)$. It implies the following.

Theorem 3.8. $\mathrm{tc}^{\mathrm{M}}(K_q) = \mathrm{tc}(K_q) = \mathrm{wgt}_{\mathrm{B}}(E_q) = \mathrm{wgt}_{\mathrm{B}}(z_q\otimes z_q) = 4$ *for all* $q \geq 2$.

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