GENERALIZED WHITEHEAD SPACES WITH FEW CELLS

Dedicated to the memory of Professor J. Frank Adams

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§0 Introduction

A topological space E is called a generalized Whitehead space (a GW space, for short) if every generalized Whitehead product on E is trivial.

The following are well known:

(0.1) E is a GW space if and only if for given maps $f : \Sigma X \to E$ and $g : \Sigma Y \to E$ there is an 'axial' map $H : \Sigma X \times \Sigma Y \to E$ such that $H|_{\Sigma X} = f$ and $H|_{\Sigma Y} = g$.

(0.2) E is a GW space if and only if for a given space W, the homotopy set $[\Sigma W, E] \cong [W, \Omega E]$ is an abelian group whose multiplication is given by the suspension structure or the loop addition.

(0.3) E is a GW space if and only if the loop space ΩE of E is homotopy abelian, that is,

$$\mu \circ T \simeq \mu$$

where $\mu : \Omega E \times \Omega E \to \Omega E$ is the loop multiplication and $T : \Omega E \times \Omega E \to \Omega E \times \Omega E$ is the switching map.

As is well known, a Hopf space always admits an axial map, and hence a Hopf space is a GW space. In other words, the notion of a GW space is a generalization of that of a Hopf space. For a sphere, however, the two notions are equivalent.

Let E be a $(q+n)$-Poincaré complex whose cells are in dimensions 0, q, n and $q+n$ with $0 < q \leq n$, for example, the total space of a spherical bundle (or fibration) over a sphere. We call such a complex *a Poincaré complex of type (q,n)*. The purpose of this paper is to show

Theorem. *If a Poincaré complex E of type (q,n) is a GW space, then $\{q, n\} \subseteq \{1, 3, 7\}$ or (q,n) = (1,2), (2,4), (3,4) or (3,5).*

The examples for these cases are as follows:

$S^q \times S^n$ for $\{q,n\} \subseteq \{1, 3, 7\}$,

$L^3(m)$ $(m \geq 1)$ for $(q,n) = (1,2)$,

$CP(3)$ for $(q,n) = (2,4)$,

S^7 for $(q,n) = (3,4)$,

$SU(3)$ for $(q,n) = (3,5)$ and

$Sp(2)$, or more generally, $E_{m\omega}$ (m $\not\equiv 2$ *mod* 4) for $(q,n) = (3,7)$. (See [**H-R**] and [**Z**] for further on $E_{m\omega}$)

This paper is organized as follows. In §1, we study a space whose cohomology is a truncated polynomial algebra of height 3 on two generators. In §2, we study a GW space whose cohomology is a truncated polynomial algebra of height 4 on one generator. In

§§3-5, we study a GW space whose cohomology is an exterior algebra on two generators. In the last section §6, we prove the main theorem.

Throughout the paper, G stands for ΩE whose loop multiplication is denoted by μ. The abbreviations $H^*(X)$ and $K^*(X)$ will be used for $H^*(X; Z_{(2)})$ and $K^*(X; Z_{(2)})$, resp. $\tilde{H}^*$ and $\tilde{K}^*$ denotes the augmentation ideal. $PH^*(X; R)$ is the submodule of primitive elements and $QH^*(X; R)$ is the quotient module of indecomposables for any coefficient ring R. $R\{a, b, c, ...\}$ means that it is an R-module with generators a, b, c,....

§1 A stable GW space

Suppose that there is a space X satisfying

$$H^*(X; Z/2) \cong Z/2^{[3]}[v_{q+1}, v_{n+1}] \quad \text{with } q \leq n \tag{1.1}$$

where the right hand side is the polynomial algebra truncated at height 3 with 2 generators v_{q+1} and v_{n+1} of degree $q + 1$ and $n + 1$, respectively.

Hence $A = H^*(X; Z/2)$ is a truncated polynomial algebra over the modulo 2 Steenrod algebra $\mathcal{A}(2)$. Then from Theorem 2.1 of [**Th1**], it follows that $q = 2^r - 1$ and $n = 2^r + 2^s - 1$ $(r - 1 \geq s \geq 0)$ or $n = 2^t - 1$ $(t \geq r)$. Again from Theorem 1.4 of [T1], it follows that

$$QA^{i+j} \subseteq Im\ Sq^i \cap Ker\ Sq^j \quad \text{if } \binom{i-1}{j} \equiv 1\ mod\ 2 \tag{1.2}$$

where QA^* indicates the quotient module of indecomposables.

Furthermore if one replaces P_2E with our X in the argument given in § 4 of [**Th2**] and the result [**Th2**, 4.5] due to Browder with the result (1.2) above which does not suppose the existence of an H-structure, one can obtain

$$q = 1,\ 3,\ 7 \text{ or } 15; \tag{1.3}$$

if X has 2-torsion in its homology, then n is even and hence $n = q + 1$ and so $v_{n+1} = Sq^1 v_{q+1}$. In particular, if $q = 15$, then $n = 16$ and $Sq^1 v_{16} = v_{17}$.

If X has no 2-torsion in its homology, then we have

$$H^*(X) \cong Z_{(2)}^{[3]}[\bar{v}_{(q+1)/2}, \bar{v}_{(n+1)/2}],$$
$$K^*(X) \cong Z_{(2)}^{[3]}[w_{(q+1)/2}, w_{(n+1)/2}].$$

Hence there is a ring isomorphism $J : H^*(X) \to K^*(X)$ given by

$$J(\bar{v}_i) = w_i, \quad \text{for } i = (q + 1)/2 \text{ and } (n + 1)/2.$$

Now the Adams operation ψ^k decomposes through Hubbuck operations $R^h_J(k)$ for an element $J(x_n)$, where x_n is in dimension n, as follows:

$$J^{-1}\psi^k J(x_n) = \sum_{i=0}^{\infty} \frac{k^n}{2^i} R^h_J(k)(x_n)$$

where $R^h_J(k)(x_n)$ increases dimension by h. The multiplicativity of Adams operations is expressed by using Hubbuck operations as the following "Cartan formula" (see [**Hu**]):

$$R^h_J(k)(v\cdot v') = \sum_{i+j=h} R^i_J(k)(v)\cdot R^j_J(k)(v')$$

Set

$$\begin{aligned} R^h &= \frac{1}{2^i}R^h_J(3), \\ P^h &= R^h_J(2) \quad \text{(the reduction } mod\ 2 \text{ of } P^h \text{ is } Sq^{2h}). \end{aligned}$$

The relation $\psi^3\psi^2 = \psi^2\psi^3$ of Adams operations is expressed by using the Hubbuck operations as follows:

$$(1.4) \qquad (3^n-1)P^n + \sum_{i=1}^{n} 3^{n-i}2^iR^iP^{n-i} = \sum_{i=1}^{n} 2^{2i}P^{n-i}R^i.$$

Furthermore, the relation $\psi^2(x_n) \equiv x_n^2\ mod\ 2$ is interpreted as

$$(1.5) \qquad \begin{aligned} P^{n+j}(x_n) &\equiv 0 \quad mod\ \ 2^{j+1} \text{ and} \\ P^n(x_n) &\equiv x_n^2 \quad mod\ \ 2 \text{ in } H^*(X). \end{aligned}$$

Note that the above formula is independent of the choice of the splitting J.

Following (1.3), we check the cases $q = 15$, 7, 3 and 1, one by one.

Consider the case $q = 15$; by (1.3) one has $n = 16$ and $Sq^1v_{16} = v_{17}$. By (1.2) one has $v_{17} \in Im\ Sq^8$, since $\binom{9-1}{8} \equiv 1\ mod\ 2$; but it contradicts $H^9(X; Z/2) = 0$. Thus $q \neq 15$.

Consider the case $q = 7$ and $n = 7 + 2^s$ with $s \leq 2$: If $s = 0$, then $Sq^1v_8 = v_9$. By (1.2) $v_9 \in Im\ Sq^4$, since $\binom{5-1}{4} \equiv 1\ mod\ 2$, but it contradicts $H^5(X; Z/2) = 0$.

If $s = 1$, then $n = 9$ and $v_{10} \in Im\ Sq^4$, since $\binom{6-1}{4} \equiv 1\ mod\ 2$; but it contradicts $H^6(X; Z/2) = 0$.

Thus $s = 2$ and then $n = 11$. We have

$$\begin{aligned} H^*(X) &\cong Z^{[3]}_{(2)}[\bar v_4, \bar v_6], \\ K^*(X) &\cong Z^{[3]}_{(2)}[w_4, w_6], \end{aligned}$$

since the homology of X is free of 2 torsion. It follows from (1.4) and (1.5) that $P^{odd} = 0$ implies $R^2 \equiv 2P^2\ mod\ 4$, and hence we obtain

$$\begin{aligned} 2P^6P^2(\bar v_4) &\equiv R^4P^4(\bar v_4) \quad mod\ 4, \\ 2\bar v_6^2 &\equiv R^4P^4(\bar v_4) \quad mod\ 4. \end{aligned}$$

Also from $R^2 \equiv 2P^2\ mod\ 4$, $P^4(\bar v_4) \equiv \lambda \bar v_4^2\ mod\ 4$ where $(\lambda,2) = 1$, and the Cartan formula, one obtains

$$0 \not\equiv 2\bar v_6^2 \equiv \lambda R^4(\bar v_4^2) \equiv \lambda \bar v_4 R^4(\bar v_4) \quad mod\ 4.$$

It is a contradiction, since the right hand side does not contribute $2\bar{v}_6^2$.

Thus $n \neq 7 + 2^s$ with $s \leq 2$.

Consider the case $q = 7$ and $n = 2^t - 1$ with $t \geq 3$: If $t = 3$, then $(q,n) = (7,7)$.

If $t = 4$, then $n = 15$. We have

$$H^*(X) \cong Z_{(2)}^{[3]}[\bar{v}_4, \bar{v}_8],$$
$$K^*(X) \cong Z_{(2)}^{[3]}[w_4, w_8].$$

Then by combining (1.4) with $P^{odd} = P^{2 \cdot odd} = 0$, one obtains that

$$2P^8 \equiv P^4 P^4 \quad mod \ \ 4 \text{ in } H^*(X). \tag{1.6}$$

From (1.5), it follows that

$$P^4(\bar{v}_8) = \alpha \bar{v}_4 \bar{v}_8, \quad \text{with } \alpha \in Z_{(2)},$$
$$P^8(\bar{v}_8) \equiv \bar{v}_8^2 \quad mod \ 2,$$
$$P^4(\bar{v}_4) \equiv \bar{v}_4^2 \quad mod \ 2$$

and hence

$$P^4(\bar{v}_4) = \lambda \bar{v}_4^2 + 2\beta \bar{v}_8, \quad \text{where } \lambda \equiv 1 \ mod \ 2.$$

Then from (1.6), it follows that

$$\begin{aligned} 2\bar{v}_8^2 &\equiv 2P^8(\bar{v}_8) \equiv P^4P^4(\bar{v}_8) \equiv \alpha P^4(\bar{v}_4\bar{v}_8) \quad mod \ 4 \\ &\equiv \alpha P^4(\bar{v}_4)\bar{v}_8 \equiv 2\alpha\beta\bar{v}_8^2 \quad mod \ 4, \end{aligned}$$

Hence $\alpha\beta \equiv 1 \ mod \ 2$. By using (1.5), however, it follows from (1.6) that

$$\begin{aligned} 0 &\equiv 2P^8(\bar{v}_4) \equiv P^4P^4(\bar{v}_4) \equiv P^4(\lambda\bar{v}_4^2 + 2\beta\bar{v}_8) \quad mod \ 4 \\ &\equiv 2\lambda\bar{v}_4P^4(\bar{v}_4) + 2\beta P^4(\bar{v}_8) \equiv 2\beta P^4(\bar{v}_8) \equiv 2\alpha\beta\bar{v}_4\bar{v}_8 \quad mod \ 4, \end{aligned}$$

which contradicts $\alpha\beta \equiv 1 \ mod \ 2$. Hence $t \neq 4$.

If $t \geq 5$, we have

$$H^*(X; Z/2) \cong Z/2^{[3]}[\bar{v}_4, \bar{v}_{2^{t-1}}].$$

Then from the main result of [**A**], it follows that

$$Sq^{2^t} \equiv \sum_{i=0}^{t-1} Sq^{2^i}\Psi_i$$

modulo the total indeterminacy which is in the image of Sq^i with $2^t > i > 0$. Now the formula gives a contradiction. In fact, the left hand side gives $Sq^{2^t}v_{2^t} \not\equiv 0 \ mod \ 2$ while the right hand side and the total indeterminacy are trivial, since

$$H^{2^{t+1}-2^i}(X) = 0 \quad \text{for } i \leq t-1.$$

It is a contradiction.

Thus $(q,n) = (7,7)$, provided that $q = 7$.

Consider the case $q = 3$ and $n = 3 + 2^s$ with $s \leq 1$: If $s = 0$, then $n = 4$ and $Sq^1 v_4 = v_5$. We have $v_5 \in Im\ Sq^2$ by (1.2), since $\binom{4-2}{2} \equiv 1\ mod\ 2$. This contradicts $H^2(X; Z/2) = 0$. Hence $s = 1$ and then $n = 5$ and $(q,n) = (3,5)$. Moreover we have $v_6 \in Im\ Sq^2$ by (1.2), since $\binom{5-2}{2} \equiv 1\ mod\ 2$.

Consider the case $q = 3$ and $n = 2^t - 1$ with $t \geq 2$: If $t = 2$, then $(q,n) = (3,3)$.

If $t = 3$, then $(q,n) = (3,7)$.

If $t \geq 4$, then we will be led to a contradiction as in the case ($q = 7$ and $n = 2^t - 1$ with $t \geq 5$).

Thus $(q,n) = (3,3)$, $(3,5)$ or $(3,7)$, provided that $q = 3$.

Consider the case $q = 1$ and $n = 1 + 2^s$ with $s \leq 0$: We have $s = 0$ and hence $(q,n) = (1,2)$. Moreover by (1.3), $Sq^1 v_2 = v_3$.

Consider the case $q = 1$ and $n = 2^t - 1$ with $t \geq 1$: If $t = 1$, then $(q,n) = (1,1)$.

If $t = 2$, then $(q,n) = (1,3)$.

If $t = 3$, then $(q,n) = (1,7)$.

If $t \geq 4$, then we will be led to a contradiction as in the case ($q = 7$ and $n = 2^t - 1$ with $t \geq 5$).

Thus $(q,n) = (1,1)$, $(1,2)$, $(1,3)$ or $(1,7)$, provided that $q = 1$.

Therefore we have shown

PROPOSITION 1.7. *If there is a space X such that*

$$H^*(X; Z/2) \cong Z/2^{[3]}[v_{q+1}, v_{n+1}]$$

with $q \leq n$, then $\{q, n\} \subseteq \{1, 3, 7\}$ or $(q,n) = (1,2)$ or $(3,5)$. Moreover if $(q,n) = (1,2)$, then $Sq^1 v_2 = v_3$; if $(q,n) = (3,5)$, then $Sq^2 v_4 = v_6$.

To apply this, we introduce the following notion.

DEFINITION. *$E \simeq S^q \cup_\alpha e^n \cup e^{q+n}$ is said to be stable if $n < 2q$.*

we get

COROLLARY 1.8. *Let E be a Poincaré complex of type (q,n). If E is a stable GW space, then $\{q, n\} \subseteq \{1, 3, 7\}$ or $(q,n) = (1,2)$, $(3,4)$ or $(3,5)$.*

Proof. By the hypotheses, $q > 1$ or $\alpha = 0$. Let Q be the subspace $S^q \cup e^n$ of E. Then, from the hypotheses, it follows that Q is desuspendable and the *mod* 2 cohomology of E is an exterior algebra except the case when $n = q + 1$ and $\alpha = m\iota_q$, m odd.

(Case 1: $n = q+1$ and $\alpha = m\iota_q$, m odd). E has the homotopy type of a $(2q+1)$-sphere at 2. Hence by the theorem of Adams [**A**], $q = 1$ or 3. Thus $(q,n) = (1,2)$ or $(3,4)$.

(Case 2: The *mod* 2 cohomology of E is an exterior algebra). There exists an axial map $\mu : Q \times Q \to E$ with axis the inclusion $Q \hookrightarrow E$. Let $Q(2)$ be the mapping cone of the Hopf construction of μ. From a direct computation using [**Th3**], we obtain that the *mod* 2 cohomology of $Q(2)$ is the polynomial algebra truncated at height 3 on the generators in dimensions $q + 1$ and $n + 1$. Hence $\{q, n\} \in \{1, 3, 7\}$ or $(q,n) = (3,5)$. This implies the corollary. QED.

§2 A GW space whose cohomology is a truncated polynomial algebra

Let E be a Poincaré complex of type (q,n) with GW space structure such that $H^*(E;Q) \cong Q[x_q]/(x_q^4)$. In this section, we will show

PROPOSITION 2.1. *If a GW space E satisfies the above condition, then $q = 2$ and $H^*(E) \cong Z_{(2)}[x_2]/(x_2^4)$.*

The remainder of this section is devoted to proving the proposition.

By the assumption on the cohomology ring, q is even. It is easy to see that

$$H^*(E; Z_{(2)}) \cong Z_{(2)}\{x_q, x_{2q}, x_{3q}\},$$

where $x_q^2 = ax_{2q}$ and $x_q x_{2q} = x_{3q}$ with $a \in Z_{(2)}$. So we have

$$E \quad \simeq_2 \quad S^q \cup_\alpha e^{2q} \cup e^{3q}, \quad \alpha \in \pi_{2q-1}(S^q)$$

Since E is a GW space, the Whitehead product of the inclusion $i : S^q \hookrightarrow E$ vanishes, and hence $i_*[\iota_q, \iota_q] = 0$ where $\iota \in \pi_q(S^q)$ is the class of the identity. For dimensional reasons $i_*[\iota_q, \iota_q]$ is already trivial in $\pi_{2q-1}(E^{[2q]}) = \pi_{2q-1}(S^q \cup_\alpha e^{2q})$. Denoting by F the homotopy fibre of i, there is a map $f : S^{2q-1} \to E$ such that $\hat{i} \circ f \simeq [\iota_q, \iota_q]$ where $\hat{i} : F \to S^q$ is the inclusion of the homotopy fibre. On the other hand,

$$F \quad \simeq_2 \quad S^{2q-1} \cup \text{ (\textit{higher dimensional cells})}$$

so that $i\mid_{S^{2q-1}} = \alpha$. If one compresses f to the lowest dimensinal cell S^{2q-1}, one obtains $[\iota_q, \iota_q] = \alpha \circ f$, where $f = \lambda\iota_{2q-1} : S^{2q-1} \to S^{2q-1}$ with $\lambda \in Z$. Thus one obtains $[\iota_q,\iota_q] = \lambda\alpha$. Taking the Hopf invariants of the both sides, one has $2 = \lambda H(\alpha)$, whence $a = \pm H(\alpha) = \pm 1$ or ± 2.

LEMMA 2.2. *$H(\alpha) = \pm 1$ and hence $q = 2$, 4 or 8.*

Proof. Suppose $H(\alpha) = \pm 2$ so that $a = \pm 2$, $\alpha = [\iota_q,\iota_q]$ and $\Sigma\alpha = 0$. This assumption leads us to a contradiction. Now the $(2q)$-skeleton of G has the following cell decomposition:

$$G^{[2q]} \quad \simeq_2 \quad S^{q-1} \cup_{[\iota_{q-1},\iota_{q-1}]} e^{2q-2} \cup e^{2q-1}.$$

Thus putting $Q = \Sigma(G^{[2q]})$, we have

$$Q \quad \simeq_2 \quad (S^q \vee S^{2q-1}) \cup_{\bar{\alpha}} e^{2q},$$

where $\bar{\alpha}$ is in $\pi_{2q-1}(S^q \vee S^{2q-1})$.

PROPOSITION 2.3. *$\bar{\alpha}$ corresponds to $(\alpha,\pm 2\iota_{2q-1})$ under the isomorphism $\pi_{2q-1}(S^q \vee S^{2q-1}) \cong \pi_{2q-1}(S^q) \oplus \pi_{2q-1}(S^{2q-1})$.*

Proof. By calculating the cohomology Serre spectral sequence associated with the path fibration $G \hookrightarrow PE \to E$, one obtains

$$\begin{aligned} H^{q-1}(G) &\cong Z_{(2)}, \\ H^{q-1+j}(G) &= 0, \quad \text{for } 1 \le j \le q-1, \\ H^{2q-1}(G) &\cong Z/2. \end{aligned}$$

Hence the composite map $p_2\bar{\alpha}$ is homotopic to $\pm 2\iota_{2q-1}$, where p_2 indicates the projection to the second factor. Moreover the natural inclusion $\lambda_1 : Q \hookrightarrow \Sigma G \hookrightarrow P^\infty G \simeq E$ induces the following commutative diagram.

$$\begin{array}{ccc} S^{2q-1} & \xrightarrow{\bar{\alpha}} & S^q \vee S^{2q-1} \\ {\scriptstyle \iota_{2q-1}}\downarrow & & \downarrow{\scriptstyle \{\iota_q,*\}} \\ S^{2q-1} & \xrightarrow{\alpha} & S^q \end{array}$$

Here both the $q-1$ and the $2q-1$ dimensional generators in $H^*(G)$ are transgressive and therefore λ_1 induces a surjection of cohomology groups in dimensions $\leq 2q$. Hence $p_1\bar{\alpha}$ is homotopic to α, where p_1 indicates the projection to the first factor. QED.

Let us recall that Q is a suspended space and E is a GW space. Hence there exists an axial map

$$\mu : Q \times Q \to E$$

with axis λ_1. So the Hopf construction of μ gives rise to a map

$$H(\mu) : \Sigma Q \wedge Q \simeq Q * Q \to \Sigma E.$$

One can see that ΣQ satisfies

$$\Sigma Q \quad \simeq_2 \quad (S^{q+1} \vee S^{2q}) \cup_{\Sigma\bar{\alpha}} e^{2q+1}.$$

By combining Proposition 2.3 with $\Sigma\alpha = 0$, one obtains that $\Sigma\bar{\alpha}$ corresponds to $(0,\pm 2\iota_{2q})$ under the isomorphism $\pi_{2q}(S^{q+1} \vee S^{2q}) \cong \pi_{2q}(S^{q+1}) \oplus \pi_{2q}(S^{2q})$. Hence we obtain

$$\Sigma Q \quad \simeq_2 \quad \Sigma S^q \vee \Sigma M^{2q},$$

where $\Sigma M^{2q} = S^{2q-1} \cup_{\pm 2\iota} e^{2q}$. Thus we obtain

$$\Sigma Q \wedge Q \quad \simeq_2 \quad \Sigma(S^q \vee M^{2q}) \wedge (S^q \vee M^{2q}),$$

which contains $\Sigma(M^{2q} \wedge M^{2q})$. We denote by $\bar{H}(\mu)$ the restriction of $H(\mu)$ to the subcomplex $\Sigma(M^{2q} \wedge M^{2q})$ and by $Q(2)$ the mapping cone of $\bar{H}(\mu)$. Then we have an exact sequence associated with it:

$$\cdots \to \tilde{H}^{*-1}(\Sigma(M^{2q} \wedge M^{2q}); Z/2) \xrightarrow{\delta} \tilde{H}^*(Q(2); Z/2) \to \tilde{H}^*(\Sigma E; Z/2) \to \cdots$$

For dimensional reasons, the sequence splits and we have

$$\begin{aligned} \tilde{H}^*(Q(2); Z/2) &\cong Z/2\{v_{q+1}, v_{2q+1}, v_{3q+1}\} \oplus Im\ \delta, \\ Im\ \delta \quad &\cong \tilde{H}^*(\Sigma(M^{2q} \wedge M^{2q}); Z/2) \\ &\cong Z/2\{x_{2q-1} \otimes x_{2q-1}, x_{2q-1} \otimes x_{2q}, x_{2q} \otimes x_{2q-1}, x_{2q} \otimes x_{2q}\} \end{aligned}$$

From [**Th3**], it follows that

$$v_{2q+1}^2 = \delta\Sigma^*(x_{2q} \otimes x_{2q}) \neq 0$$

and hence $0 \neq Sq^{2q+1}v_{2q+1}$. Let us recall the Adem relation

$$Sq^q Sq^{q+1} = Sq^{2q+1} + \binom{q-1}{q-2} Sq^{2q}Sq^1 + ... + \binom{\frac{q}{2}}{0} Sq^{3q/2+1}Sq^{q/2},$$

for q even. For j with $1 \leq j \leq q/2$, we have

$$deg\ Sq^j v_{2q+1} = 2q + j + 1 < 3q + 1 < 4q,$$

which implies $Sq^j v_{2q+1} = 0$ and hence $Sq^{q+1}v_{2q+1} \neq 0$. The Adem relation $Sq^{q+1} = Sq^1 Sq^q$ (q even) implies that $Sq^q v_{2q+1} \neq 0$ and therefore $Sq^q v_{2q+1} = v_{3q+1}$. Hence $Sq^1 v_{3q+1} \neq 0$ where $deg\ Sq^1 v_{3q+1} = 3q + 2 \leq 4q$. Thus $3q + 2 = 4q$ and hence $q = 2$.

Even when $q = 2$, one has

$$Sq^1 v_{3q+1} = \delta\Sigma^*(x_{2q-1} \otimes x_{2q-1})$$

and hence

$$\begin{aligned} 0 &= Sq^1 Sq^1 v_{3q+1} \\ &= \delta\Sigma^* Sq^1(x_{2q-1} \otimes x_{2q-1}) \\ &= \delta\Sigma^*(x_{2q} \otimes x_{2q-1} + x_{2q-1} \otimes x_{2q}) \\ &\neq 0 \end{aligned}$$

which is a contradiction. This implies that $\Sigma\alpha \neq 0$. Thus $H(\alpha) = \pm 1$ and hence $q = 2$, 4 or 8. QED.

According to [**To**], $[\iota_q, \iota_q] = 2\alpha$ holds only when $q = 2$. Thus we have $H^*(E) \cong Z_{(2)}[x_2]/(x_2^4)$.

Remark. $H^*(CP^3) \cong Z_{(2)}[x_2]/(x_2^4)$.

§3 A GW space whose cohomology is an exterior algebra

Throughout the section let E be a Poincaré complex of type (q,n) with GW space structure such that

$$H^*(E) = \wedge(x_q, x_n), \quad 1 \leq q < n$$

If $q = 1$, then the GW space structure inherits the universal covering space $\tilde{E}$ of E, which has the homotopy type of S^n. Let us recall that a sphere is a GW space if and only if it is an H-space. Hence $n = 3$ or 7.

We will prove that both q and n are odd integers, even when $q > 1$.

First we show

(3.1) q is odd.

Consider the cohomology Serre spectral sequence with $Z_{(2)}$ coefficients associated with the path fibration $G \hookrightarrow PE \to E$. Since the element $x_q \in H^q(E)$ is in the image of the transgression, we have $0 \neq \sigma^* x_q \in H^{q-1}(G) \cong Z_{(2)}$, where $\sigma^* : H^*(E) \to H^{*-1}(G)$ is the cohomology suspension. So $u_{q-1} = \sigma^* x_q$ is transgressive, and hence is primitive. Thus the element $\Sigma^* u_{q-1} \in H^q(\Sigma G)$ is extendable to P^2G and the extension is given by the image of x_q under the induced map of the composite map

$$\lambda_2 : P^2G \hookrightarrow P^\infty G \simeq E$$

since $\sigma^* x_q$ is represented by a loop map whose delooping is given by x_q. Hence we obtain

$$\bar{x}_q^2 = 0 \quad in \quad H^*(P^2G).$$

Now we recall that the element $\bar{x}_q^2$ is given by $\bar{x}_q^2 = \pm\, \delta_2 \Sigma^*(u_{q-1} \otimes u_{q-1})$ where δ_2 is the operation given in [**Th2**]. So it follows from the triviality of $\bar{x}_q^2$ that $u_{q-1} \otimes u_{q-1}$ is in the image of $\bar{\mu}^* = \mu^* - p_1^* - p_2^*$:

$$\bar{\mu}^* = \mu^* - p_1^* - p_2^* : \tilde{H}^*(G) \to \tilde{H}^*(G) \otimes \tilde{H}^*(G).$$

So the relation (0.1) implies that the element $u_{q-1} \otimes u_{q-1}$ is T^*-invariant where T is the switching map. On the other hand, $T^*(u_{q-1} \otimes u_{q-1}) = -u_{q-1} \otimes u_{q-1}$ if q is even. Hence it cannot be T^*-invariant, since it is a generator of $\tilde{H}^{2(q-1)}(G \wedge G) \cong \tilde{H}^{q-1}(G) \otimes \tilde{H}^{q-1}(G)$ and not of order 2. Thus q have to be odd.

Next we show

(3.2) $\qquad n$ is odd.

Suppose that n is even. Then $n-1$ is odd and is not divisible by $q-1$, which is known to be even. It follows that $u_{n-1} = \sigma^* x_n$ is non trivial and indecomposable. As in the case with q, the element $\Sigma^* u_{n-1}$ is extendable over P^2G. Denoting by $\bar{x}_n$ the extended element, we have

$$\bar{x}_n^2 = 0 \quad \text{in } H^*(P^2G).$$

It means that the element $u_{n-1} \otimes u_{n-1}$ is in the image of $\bar{\mu}^*$. On the other hand $u_{n-1} \otimes u_{n-1}$ belongs to $\tilde{H}^{2n-2}(G \wedge G)$, which contains the direct summand $\tilde{H}^{n-1}(G) \otimes \tilde{H}^{n-1}(G) \cong Z_{(2)}$ generated by $u_{n-1} \otimes u_{n-1}$, which implies that $u_{n-1} \otimes u_{n-1} \notin Im\ \bar{\mu}^*$; it is a contradiction. This implies that n is odd.

Thus we have shown

Proposition 3.3. *If E is a GW space with $H^*(E) = \wedge(x_q, x_n)$, then both q and n are odd. If in addition $q = 1$, then $n = 3$ or 7.*

In the remainder of this section, we assume that $q > 1$. Since q and n are odd, we may assume that $q + 1 < n$.

Now we choose an inclusion map $j : S^q \to E$ such that $j^* x_q$ is a generator of $H^q(S^q) \cong Z_{(2)}$ (since we do not assume that $S^q \hookrightarrow E \to S^n$ is a fibration in this section). Denote by

F the homotopy fibre of j, that is, $F \to S^q \to E$ is a Serre fibration). Then by the Serre spectral sequence one sees

$$H^*(F) \cong H^*(\Omega S^q)$$

Similarly the Serre spectral sequence of the fibration $\Omega S^q \hookrightarrow G \to F$ collapses and hence

(3.4) $$H^*(G) \cong H^*(\Omega S^q) \otimes H^*(\Omega S^n) \quad \text{as modules,}$$

in particular

(3.4') $$H^*(G) \cong H^*(\Omega S^q) \qquad \text{for } * < n-1.$$

Here a system of ring generators of $H^*(\Omega S^q)$ is given by

(3.5) $$u_{q-1} = \gamma_1 u_{q-1}, \gamma_2 u_{q-1}, ..., \gamma_j u_{q-1}, ...,$$

where $j \geq 1$ and $u_{q-1} = \sigma^* x_q$.

One obtains from (3.4) the following extension of bicommutative biassociative Hopf algebras:

$$Z_{(2)} \to H^*(\Omega S^n) \to H^*(G) \to H^*(\Omega S^q) \to Z_{(2)}$$

PROPOSITION 3.6. *The following is a commutative diagram of the exact sequences:*

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & \longrightarrow & PH^*(\Omega S^n; Z/2) & \longrightarrow & QH^*(\Omega S^n; Z/2) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & P(Z/2(\xi H^*(G; Z/2))) & \longrightarrow & PH^*(G; Z/2) & \longrightarrow & QH^*(G; Z/2) \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & PH^*(\Omega S^q; Z/2) & \longrightarrow & QH^*(\Omega S^q; Z/2) \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

where the element $\tilde{u}_{n-1}$ (and $\tilde{u}_{q-1}$) the modulo 2 reduction of u_{n-1} (and u_{q-1}, resp.) generates $PH^*(\Omega S^n; Z/2) \cong Z/2$ (and $PH^*(\Omega S^q; Z/2) \cong Z/2$, resp.).

It follows from (3.5) that the first non-trivial relation can occur in degree $n-1$ only when there is a non-negative integer r such that

$$n - 1 = 2^{r+1}(q-1).$$

Then the relation is

(3.7) $$\tilde{u}_{n-1} = (\gamma_{2^r} \tilde{u}_{q-1})^2$$

where $\tilde{u}_\ell$ is the modulo 2 reduction of u_ℓ for $\ell = q-1$ and $n-1$. Thus it follows that $n \equiv 1$ *mod* 4.

THEOREM 3.8. *(i) If $n \equiv 1$ mod 4, then $\tilde{x}_n = Sq^2\ \tilde{x}_q$ and $(q,n) = (3,5)$,*
(ii) $q \equiv 3$ mod 4,
where $\tilde{x}_\ell$ is the modulo 2 reduction of x_ℓ for $\ell = q$ and n.

The remainder of this section will be devoted to proving this theorem. First in the general situation, we will construct a space and compute its cohomology ring. The cell structure of the n-skeleton of G is as follows:

$$G^{[n]} \quad \simeq_2 \quad (\Omega S^q)^{[n]} \cup e^{n-1}.$$

Thus putting $Q = \Sigma(G^{[n]})$, we have

$$\begin{aligned} Q \quad &\simeq_2 \quad (\bigvee_{i=1}^{[\frac{n-1}{q-1}]} S^{i(q-1)+1}) \cup e^n \\ \Sigma Q \quad &\simeq_2 \quad (\bigvee_{i=1}^{[\frac{n-1}{q-1}]} S^{i(q-1)+2}) \cup e^{n+1} \end{aligned}$$

The module $QH^*(E)$ is mapped injectively into $H^*(Q)$ by the induced homomorphism of the canonical inclusion

$$\lambda_1 : Q \subset \Sigma G \subset P^\infty G \simeq E.$$

In fact, as was already seen, $PH^*(G) \cong Z_{(2)}\{u_{q-1}, u_{n-1}\}$ with u_i transgressive, and λ_1^* gives rise to the cohomology suspension. Thus we obtain

$$Im\ (\Sigma\lambda_1)^* \cong Z_{(2)}\{v_{q+1}, v_{n+1}\}$$

which is a direct summand of $\tilde{H}^*(\Sigma Q)$. Hence we have

$$\tilde{H}^*(\Sigma Q) \cong Im\ (\Sigma\lambda_1)^* \oplus D,$$

where D is the module generated by elements $\gamma_i u_{q-1}$ with $i \geq 2$. Since Q is a suspension space, there exists an axial map

$$\mu : Q \times Q \to E$$

with axis λ_1. So the Hopf construction of μ gives rise to a map

$$H(\mu) : \Sigma Q \wedge Q \simeq Q * Q \to \Sigma E.$$

We denote by $Q(2)$ the mapping cone of $H(\mu)$, so that we have a cofibre sequence

$$\Sigma E \xrightarrow{j} Q(2) \to \Sigma Q \wedge \Sigma Q. \tag{3.9}$$

The elements x_q, $x_n \in \tilde{H}^*(E)$ are primitive with respect to μ in the sense of Thomas, since $\tilde{H}^{odd}(Q \wedge Q) = 0$. Hence we have

$$\begin{aligned} \tilde{\mu}^*(x_i) \quad &= 0 \qquad \text{for } i = q,n, \\ \tilde{\mu}^*(x_q x_i) &= \lambda_1^* x_q \otimes \lambda_1^* x_n - \lambda_1^* x_n \otimes \lambda_1^* x_q. \end{aligned}$$

So the image of j^* induced by the inclusion $j : \Sigma E \hookrightarrow Q(2)$ are given by

$$Im\ j^* \cong Z_{(2)}\{\Sigma^* x_q, \Sigma^* x_n\}.$$

Also the image and the kernel of δ induced by the collapsing map $Q(2) \to \Sigma Q \wedge \Sigma Q \cong \Sigma^4(G^{[n]} \wedge G^{[n]})$ is given by

$$\begin{aligned} &Im\ \delta \cong \delta(\Sigma^4)^* Z_{(2)}\{u_i \otimes u_j; i, j = q-1 \ or \ n-1\} \oplus S_2, \\ &Ker\ \delta \cong (\Sigma^4)^* Z_{(2)}\{u_{q-1} \otimes u_{n-1} - u_{n-1} \otimes u_{q-1}\} \end{aligned} \tag{3.10}$$

where $S_2 \cong \delta(D \otimes \tilde{H}^*(\Sigma Q)) \oplus \delta(\tilde{H}^*(\Sigma Q) \otimes D)$. Therefore by (3.9), we obtain the following short exact sequence :

$$0 \to Im\ \delta \to \tilde{H}^*(Q(2)) \to Z_{(2)}\{\Sigma^* x_q, \Sigma^* x_n\} \to 0$$

Thus denoting by v_{i+1} the extension of $\Sigma^* x_i$ over $Q(2)$, $i = q$ and n, we obtain the following ring isomorphisms by virtue of [**Th3**]:

$$\begin{aligned} &H^*(Q(2)) \cong Z_{(2)}^{[3]}[v_{q+1}, v_{n+1}] \oplus S_2, \\ &\tilde{H}^*(Q(2)) \cdot S_2 = 0 \end{aligned} \tag{3.11}$$

where $v_{i+1} \cdot v_{j+1} = \delta(\Sigma^4)^*(u_{i-1} \otimes u_{j-1})$

Remark that these results are independent of the choice of v_{q+1} and v_{n+1}.

Proposition 3.12. *(1) $Q(2)$ has no torsion and hence $Sq^1 \tilde{H}^*(Q(2); Z/2) = 0$*

(2) $\mathcal{A}(2)(Z/2\{\tilde{v}_{q+1}, \tilde{v}_{n+1}\}) \subset Z/2^{[3]}[\tilde{v}_{q+1}, \tilde{v}_{n+1}] \oplus (S_2 \otimes Z/2)$

(3) $\mathcal{A}(2)(Im\ \delta \otimes Z/2) \subseteq Im\ \delta \otimes Z/2$, where $\tilde{v}_\ell$ is the modulo 2 reduction of v_ℓ for $\ell = q+1$ and $n+1$.

The following two propositions imply Theorem 3.8.

Proposition 3.13. *If $n \equiv 1 \bmod 4$, then $\tilde{x}_n = Sq^2 \tilde{x}_q$ and $(q,n) = (3,5)$*

Proof. By (3.11), $H^*(Q(2); Z/2)$ has a direct summand $Z/2^{[3]}[\tilde{v}_{q+1}, \tilde{v}_{n+1}]$, where $\tilde{v}_\ell$ is the modulo 2 reduction of v_ℓ for $\ell = q+1$ and $n+1$. If $n = 4m+1$ for some $m \geq 1$, we have

$$0 \neq \tilde{v}_{n+1}^2 = Sq^{4m+2} \tilde{v}_{n+1}$$

where $Sq^{4m+2} = Sq^2 Sq^{4m} + Sq^1 Sq^{4m} Sq^1$.

So we have that $\tilde{v}_{n+1}^2 \in Im\ Sq^2$, since $Sq^1 = 0$ on $H^*(Q(2); Z/2)$. Hence we have $\tilde{v}_{n+1}^2 = \delta(\Sigma^4)^*(\tilde{u}_{n-1} \otimes \tilde{u}_{n-1}) \in Sq^2\ Im\ \delta$, where $\tilde{u}_\ell$ is the modulo 2 reduction of u_ℓ, $\ell = q+1$ and $n+1$, for dimensional reasons.

Hence we obtain that $\tilde{u}_{n-1} \otimes \tilde{u}_{n-1} \in Im\ Sq^2$ in $\tilde{H}^*(G^{[n]} \wedge G^{[n]}; Z/2)$ modulo the kernel of $\delta \otimes Z/2$.

By (3.10), we have $Z/2\{\tilde{u}_{n-1} \otimes \tilde{u}_{n-1}\} \cap Ker\ \delta = 0$, which implies that $\tilde{u}_{n-1} \otimes \tilde{u}_{n-1} \in Im\ Sq^2$. Thus we obtain that $\tilde{u}_{n-1} \in Im\ Sq^2$ in $\tilde{H}^*(G^{[n]}; Z/2)$. There are two cases:

If $\tilde{u}_{n-1}$ is decomposable, we have $\tilde{u}_{n-1} = (\gamma_j \tilde{u}_{q-1})^2$ for some $j > 0$ by Proposition 3.6, and so $\gamma_{2^r}\tilde{u}_{q-1} \in Im\ Sq^2$. This relation holds in $\tilde{H}^*((\Omega S^q)^{[n]}; Z/2)$, since $deg\ \gamma_{2^r}\tilde{u}_{q-1} < n-1$. This contradicts that $\Sigma\Omega S^q$ is a bouquet of spheres. Thus $\tilde{u}_{n-1}$ is indecomposable. Therefore there exists a non-negative integer r such that $Sq^2\gamma_{2^r}\tilde{u}_{q-1} = \tilde{u}_{n-1}$.

Comparing the degrees of both sides, we have $2 + 2^r(q-1) = n-1 = 4m$, whence one has $r = 0$, since $q-1$ is even by Proposition 3.3. This implies that $Sq^2\tilde{u}_{q-1} = \tilde{u}_{n-1} \neq 0$ and hence $n = q+2 > 4$ and $Q \simeq_2 S^q \cup e^n$. Then the *mod* 2 cohomology of $Q(2)$ satisfies the condition given in §1. Hence from Corollary 1.8, it follows that $(q,n) = (q,q+2)$ have to be (3,5).

Proposition 3.14. *$q \equiv 3$ mod 4.*

Proof. Similarly we have $\tilde{v}_{q+1}^2 \not\equiv 0$ in $\tilde{H}^*(Q(2); Z/2)$. If $q \equiv 1$ *mod* 4, then one has $\tilde{v}_{q+1}^2 \in Im\ Sq^2$. Also $deg\ \tilde{v}_{q+1}^2 - 2 = 2q \equiv 2$ *mod* 4. If $n \equiv 1$ *mod* 4, then $q = 3 \not\equiv 1$ *mod* 4, which is a contradiction. So $n \equiv 3$ *mod* 4, whence $2q \neq n+1$. Thus, one has that $\tilde{v}_{q+1}^2 \in Sq^2\ Im\ \delta$. By an argument similar to that given in the proof of Proposition 3.13, we obtain that $\tilde{u}_{q-1} \otimes \tilde{u}_{q-1} \in Im\ Sq^2$ in $\tilde{H}^*(G^{[n]} \wedge G^{[n]}; Z/2)$. This implies that $\tilde{u}_{q-1} \in Im\ Sq^2$ in $\tilde{H}^*(G^{[n]})$ while $G^{[n]}$ is $(q-2)$ connected. It is a contradiction and completes the proof of the proposition. QED.

§4 Unstable GW spaces

Let E be a GW space such that $\tilde{H}^*(E; Z/2) = \wedge(x_q, x_n)$ with $1 \leq q < n$.

Proposition 4.1. *E has the homotopy type of $S^q \cup_\alpha e^n \cup_\beta e^{n+q}$ where $\alpha \in \pi_{n-1}(S^q)$ and $\beta \in \pi_{n-q-1}(S^q \cup S^n)$.*

Definition. *$E \simeq S^q \cup_\alpha e^n \cup e^{q+n}$ is said to be unstable if $2q \leq n$.*

By Proposition 3.3, we have that both q and n are odd integers. So $2q < n$, if E is unstable.

We will show

Theorem 4.2. *If the above E is an unstable GW space, then (q,n) is one of the following: (1,3), (1,7), (3,7), (3,11) or (7,15).*

The remainder of the section is devoted to proving the theorem.

Let $j : S^q \to E$ be the inclusion of the bottom sphere S^q. Consider the map $\{j,j\} : S^q \vee S^q \to E$. We have that the Whitehead product $[j,j]$ is homotopic to zero, as E is a GW space. Hence the map $\{j,j\}$ is extendable over $S^q \times S^q \to E$. By the assumption that $2q < n$, the image of μ is compressible into S^q so that S^q is an H-space, whence $q = 1$, 3 or 7 by the theorem of Adams [**A**].

[The case $q = 1$] The universal covering space $\tilde{E}$ of E is easily seen to be a GW space having the same homotopy type as S^n, which then becomes an H-space. Again by the theorem of [**A**], $n = 1$, 3 or 7. Omitting the case $n = 1$, we have $(q,n) = (1,3)$ or $(1,7)$.
[The case $q = 3$ or 7] Put $\varepsilon = 1$ or 3 according as $q = 3$ or 7, i.e. $\varepsilon = \frac{1}{2}(q-1)$. If $n \equiv 1$ *mod* 4, we obtain, by Theorem 3.8, that $(q,n) = (3,5)$, which contradicts $n > 2q$. Hence $n \equiv 3$ *mod* 4. If the element $u_{n-1} = \sigma^* x_n$ of $PH^{n-1}(E; Z/2)$ is decomposable in $H^*(\Omega E; Z/2)$, then by Proposition 3.6 it is in the image of $\xi : H^*(\Omega E; Z/2) \to H^*(\Omega E; Z/2)$, which is impossible by the fact that $n-1 \equiv 2$ *mod* 4. Thus u_{n-1} is indecomposable in $H^*(\Omega E; Z/2)$.

PROPOSITION 4.3. *If Sq^2 is non-trivial on $H^*(\Omega E; Z/2)$, then $n = 2^{i+2}\varepsilon + 3$ for some $i \geq 0$.*

Proof. Put $u_{q-1} = \sigma^* x_q$ and $u_{n-1} = \sigma^* x_n$. Let $\omega \in H^*(\Omega E; Z/2)$ be an element of the lowest degree such that $Sq^2\omega \neq 0$. Then $S_q^2\omega$ is primitive, and so $S_q^2\omega = u_{q-1}$ or u_{n-1}. It follows from $H^{q-3}(\Omega E; Z/2) = 0$, that $Sq^2\omega = u_{n-1}$. Thus ω is a generater of lower degree than $n-1$, whence one can express it as $\omega = \gamma_2^{i+1} u_{q-1}$ for some $i \geq 0$ (, since $\gamma_1 u_{q-1} = u_{q-1}$ is not mapped to u_{n-1} by Sq^2). Comparing the degrees we have $2^{i+1}(q-1) + 2 = n-1$, and so $n = 2^{i+1}\varepsilon + 3$ for some $i \geq 0$.

PROPOSITION 4.4. *If $Sq^2 = 0$ on $H^*(\Omega E; Z/2)$, then $Sq^{2^i} = 0$ on $H^*(\Omega E; Z/2)$ for any $i \geq 0$.*

Proof. Suppose $Sq^1 = \ldots = Sq^{2^{j-1}} = 0$ and $Sq^{2^j} \neq 0$ on $H^*(\Omega E; Z/2)$. By assumption, we have $j \geq 2$. As in the proof of Proposition 4.3, one can conclude that

$$Sq^{2^j} \gamma_{2^{i+1}} u_{q-1} = u_{n-1} \quad \text{for some } i \geq 0,$$

(since $\gamma_1 u_{q-1} = u_{q-1}$ is not mapped to u_{n-1} by any squaring operation from the fact that $2(q-1) < n-1$). Comparing the degrees one has $2^{i+1}(q-1) + 2^j = n-1$; it gives $n-1 \equiv 0$ *mod* 4 after reducing *mod* 4, since $j \geq 2$ and $q-1 \equiv 0$ *mod* 2. This contradicts $n \equiv 3$ *mod* 4. QED.

Quite similarly one obtains

PROPOSITION 4.5. *If $u_{n-1} \in Im\ Sq^{2^j}$, then $j = 1$.*

We will discuss the two cases, whether Sq^2 acts trivially or not, by using the methods given in §3.

THEOREM 4.6. *If $Sq^2 = 0$ on $H^*(\Omega E; Z/2)$, then $(q,n) = (3,7)$.*

Proof. It follows from Proposition 4.4 that any *mod* 2 Steenrod operations act trivially on $H^*(\Omega E; Z/2)$. Let $Q(2)$ be as in §3, then we have

$$H^*(Q(2); Z_{(2)}) \cong Z_{(2)}^{[3]}[v_{q+1}, v_{n+1}] \oplus S_2,$$

By (3.10), (3.11), Proposition 3.12 and Proposition 4.4, we get

PROPOSITION 4.7. *If $v_{n+1}^2 \in Im\ \theta$ in the algebra $H^*(Q(2); Z_{(2)})$ for some $\theta \in \mathcal{A}(2)$ and if $Sq^2 = 0$ on $H^*(\Omega E; Z/2) = 0$, then $\theta = Sq^{n+1}$.*

Now we will examine the decomposition of $Sq^{2^{k+1}}$ $(k \geq 0)$ through secondary operations on the space $X = Q(2)$, which is the main result in [**A**]. If $n+1$ is not a power of 2, then by the Adem relation

$$0 \neq v_{n+1}^2 = Sq^{n+1}(v_{n+1}) = \sum_i a_i b_i(v_{n+1}),\ 0 < deg\ a_i < n+1$$

which contradicts Proposition 4.7.

When $n = 2^{k+4} - 1$, $k \geq 0$, there holds

$$0 \neq v_{n+1}^2 = Sq^{n+1}(v_{n+1}) = \sum_{i,j} a_{ij}\Phi_{ij}(v_{n+1}),\ 0 < deg\ a_{ij} < n+1$$

modulo $a_{ijk}Q^{2n+2-l}(i,j,k)(Q(2);Z/2)$ where $0 < l(i,j,k) = deg\ a_{ijk} < n+1$. Thus the element v_{n+1}^2 belongs to the image of a certain Steenrod operation a with $0 < deg\ a < n+1$. This also contradicts Proposition 4.7. So, if $n+1 = 2^k$, then $k =$ 0,1,2 or 3.

The equation $2q = 4\varepsilon + 1 < n = 2^k - 1$ implies that $n = 7$ if $q = 3$ and that n does not exist if $q = 7$. Thus Theorem 4.6 is proved. QED

THEOREM 4.8. *If* $Sq^2 \neq 0$ *on* $\tilde{H}^*(\Omega E; Z/2)$*, then (q,n) = (3,7), (3,11) or (7,15).*

Proof. It follows from Proposition 4.5 that $n = 2^{i+2}\cdot\varepsilon + 3$ for some $i \geq 0$. If $i = 0$, then (q,n) = (3,7) or (7,15). We assume $i \geq 1$. Then $n+1 = 2^{i+2}\cdot\varepsilon + 4 \equiv 4\ mod\ 8$. So by the Adem relation we have

$$\begin{aligned} Sq^4 Sq^{2^{i+2}\cdot\varepsilon} &= Sq^{n+1} + Sq^{2^{i+2}\cdot\varepsilon+2}Sq^2 + Sq^{2^{i+2}\cdot\varepsilon+3}Sq^1 \\ &= Sq^{n+1} + Sq^{2+2^{i+2}\cdot\varepsilon}Sq^2 + Sq^3 Sq^{2^{i+2}\varepsilon}Sq^1 \end{aligned}$$

Again by (3.10), (3.11) and Proposition 3.12, we obtain

$$Sq^2 v_{n+1} \in \delta(\Sigma^4)^*\tilde{H}^*(\Omega S^q \wedge \Omega S^q) \subseteq \delta(\Sigma^4)^* H^*(\Omega E \wedge \Omega E),$$

since $deg\ Sq^2 v_{n+1} = 2 + deg\ v_{n+1} = 4 + deg\ u_{n-1}$ $(= 4 + 2^{i+2}\cdot\varepsilon + 2)$. Thus the folowing conditions are necessary for $Sq^{2+2^{i+2}\cdot\varepsilon}Sq^2 v_{n+1}$ to contribute to $v_{n+1}^2 = \delta(\Sigma^4)^*(u_{n-1} \otimes u_{n-1})$: There are elements $\hat{u}_1$ and $\hat{u}_2$ such that

$$\begin{aligned} &Sq^2 v_{n+1} = \delta\Sigma^4(\hat{u}_1 \otimes \hat{u}_2) + \text{ other terms} \\ &Sq^{2+2^{i+2}\varepsilon}(\hat{u}_1 \otimes \hat{u}_2) = u_{n-1} \otimes u_{n-1} + \text{ other terms} \end{aligned}$$

However, we have $deg\ \hat{u}_1 \otimes \hat{u}_2 = 2 + 2^{i+2}\cdot\varepsilon$ since $deg\ u_{n-1} = 2 + 2^{i+2}\cdot\varepsilon$. Therefore $Sq^{2+2^{i+2}\cdot\varepsilon}(\hat{u}_1 \otimes \hat{u}_2) = \hat{u}_1^2 \otimes \hat{u}_2$, which contradicts the indecomposability of u_{n-1}. Thus, since $Sq^{2+2^{i+2}\cdot\varepsilon}Sq^2 v_{n+1}$ does not contribute to v_{n+1}^2, one of elements $Sq^4 Sq^{2^{i+2}\cdot\varepsilon}v_{n+1}$ has to do so in its place. Here we remark that

$$Sq^{2^{i+2}\varepsilon}v_{n+1} \in Im\ \delta$$

So the following two cases can occur:

(1) $$\begin{aligned} &Sq^{2^{i+2}\cdot\varepsilon}v_{n+1} = \delta\Sigma^4(\gamma_{2^{i_1}}u_{q-1} \otimes \gamma_{2^{i_2}}u_{q-1}) + \text{ other terms} \\ &Sq^4(\gamma_{2^{i_1}}u_{q-1} \otimes \gamma_{2^{i_2}}u_{q-1}) = u_{n-1} \otimes u_{n-1} + \text{ other terms} \end{aligned}$$

(2) $$\begin{aligned} &Sq^{2^{i+2}\cdot\varepsilon}v_{n+1} = \delta\Sigma^4(\gamma_{2^{i_1}}u_{q-1} \otimes u_{n-1}) + \text{ other terms} \\ &Sq^4\gamma_{2^{i_1}}u_{q-1} = u_{n-1} + \text{ other terms} \end{aligned}$$

But the latter case does not occur by Proposition 4.3. So we obtain

(a) $$Sq^{2^{i+2}\varepsilon}v_{n+1} = \delta\Sigma^3(\gamma_{2^{i_1}}u_{q-1}\otimes\gamma_{2^{i_1}}u_{q-1}) + \text{ other terms}$$

(b) $$Sq^2\gamma_{2^{i_1}}u_{q-1} = u_{n-1} + \text{ other terms}$$

Comparing the degrees we obtain $i_1 = i$ from (b). We also have $\gamma_{2^{i_1}}u_{q-1} \in \tilde{H}^*(\Omega S^q) \subseteq \tilde{H}^*(\Omega E)$, as $deg\ \gamma_{2^i}u_{q-1} < n-1$. Hence the element $\gamma_{2^i}u_{q-1}$ does not belong to the image of any squaring operations on $\tilde{H}^*(\Omega E; Z/2)$.

Now we divide the arguments into the two cases, $\varepsilon = 1$ and $\varepsilon = 3$.

[The case $\varepsilon = 3$] The Adem relation

$$Sq^{2^{i+2}\varepsilon} = Sq^{2^{i+3}+2^{i+2}} = \sum_{t=0}^{i+2} Sq^{2^t}a_t, a_t \in A(2)$$

implies that $\gamma_{2^i}u_{q-1}\otimes\gamma_{2^i}u_{q-1} \in Sq^{2^t}a_t$ for some $0 \le t \le i+2$. On the other hand, one can deduce from $a_t(v_{n+1}) \in Im\ \delta$ that $\gamma_{2^i}u_{q-1}\otimes\gamma_{2^i}u_{q-1} \in Im\ Sq^{2^t}$ in $H^*(\Omega E\wedge\Omega E; Z/2)$ for some t, which contradicts the fact that $\gamma_{2^i}u_{q-1}$ is not in the image of any squaring operations.

[The case $\varepsilon = 1$] If $i = 1$, then $(q,n) = (3,11)$. Suppose $i \ge 2$. By [**A**] $Sq^{2^{i+2}}$ is decomposable through secondary operations, that is, the following holds

$$Sq^{2^{i+2}}(v_{n+1}) = \sum_{i,j} a_{ij}\Phi_{ij}(v_{n+1}), 0 < deg\ a_{ij} < 2^{i+2}$$

modulo the total indeterminacy $a_{ijk}Q^{2^{i+3}+4-l(i,j,k)}(Q(2); Z/2)$, $0 < l(i,j,k) = deg\ a_{ijk} < 2^{i+2}$.

This leads us to a contradiction similarly to the case when $\varepsilon = 3$.

§5 The non-existence of types (3,11) and (7,15)

PROPOSITION 5.1.

$$(q,n) \neq (3,11)$$

Proof. If $(q,n) = (3,11)$, then $E \simeq S^3\cup_\alpha e^{11}\cup_\beta e^14$ where $\alpha \in \pi_{10}(S^3) \cong$ Z/15. So $E \simeq_2 (S^3\vee S^{11})\cup_\beta e^{14}$. Since $Q = S^3\vee S^{11}$ is desuspendable, the Whitehead product $[i,i]$ of the inclusion $i: Q \hookrightarrow E$ vanishes by assumption. So the map $\{i,i\}: Q\vee Q \to E$ is extendable over $Q\times Q$. We denote the extension by $\mu: Q\times Q \to E$. If we put $Q(2) = C_{H(\mu)}$, the cofibre of the Hopf construction of μ, then $Q(2)$ satisfies the condition of §1. It gives a contradiction, and so $(q,n) \neq (3,11)$. QED

PROPOSITION 5.2.

$$(q,n) \neq (7,15)$$

Proof. Suppose $(q,n) = (7,15)$ so that $E \simeq_2 S^7\cup_\alpha e^{15}\cup e^{22}$. Then we have

$$H^*(E) \cong \Lambda(x_7, x_{15})$$
$$K^*(E) \cong \Lambda(\xi_7, \xi_{15}).$$

The 15-skeleton of $G = \Omega E$ is given by

$$G^{[15]} \simeq_2 S^6 \cup_{[\iota_6,\iota_6]} e^{12} \cup e^{14}.$$

Now we put $Q = \Sigma(G^{[15]})$; then

$$Q \simeq_2 (S^7 \vee S^{13}) \underset{\bar{\alpha}}{\cup} e^{15}, \text{ where } \bar{\alpha} \in \pi_{14}(S^7 \vee S^{13}) \cong \pi_{14}(S^7) \oplus \pi_{14}(S^{13});$$
$$\Sigma Q \simeq_2 (S^8 \vee S^{14}) \underset{\Sigma\bar{\alpha}}{\cup} e^{16}.$$

The generators of $H^*(E)$ and $K^*(E)$ are mapped monomorphically to $H^*(Q)$ and $K^*(Q)$, respectively, by the induced homomorphism of the canonical inclusion $\lambda_1 : Q \subset \Sigma G \subset P^\infty G \simeq E$. In fact, as was already seen, $PH^*(G) \cong Z_{(2)}\{u_6, u_4\}$ with u_i trangressive, and λ_1^* gives rise to the cohomology suspension. Thus we obtain

$$Im\ (\Sigma\lambda_1)^* \cong Z_{(2)}\{v_8, v_{16}\} \subseteq H^*(\Sigma Q) \cong Z_{(2)}\{v_8, v_{14}, v_{16}\},$$
$$Im\ (\Sigma\lambda_1)^* \cong Z_{(2)}\{w_4, w_8\} \subseteq K^*(\Sigma Q) \cong Z_{(2)}\{w_4, w_7, w_8\}.$$

Then the Adams operation ψ^k in $K^*(\Sigma Q)$ is given by

$$\begin{aligned} \psi^k w_4 &= k^4 w_4 + a(k) w_8 \\ \psi^k w_7 &= k^7 w_7 + b(k) w_8 \\ \psi^k w_8 &= k^8 w_8 \end{aligned} \tag{5.3}$$

Since Q is a suspended space and since E is a GW space, there exists an axial map

$$\mu : Q \times Q \to E$$

with axis λ_1. We denote by $Q(2)$ the mapping cone of the Hopf construction $H(\mu)$ of the map μ so that we have a cofibre sequence

$$\Sigma E \overset{j}{\hookrightarrow} Q(2) \to \Sigma Q \wedge \Sigma Q. \tag{5.4}$$

The elements x_7, $x_{15} \in H^*(E)$ are primitive with respect to μ in the sense of Thomas as $H^{11}(Q \wedge Q) = H^{15}(Q \wedge Q) = 0$. Hence we have

$$\bar{\mu}^*(x_i) = 0 \quad \text{for } i = 7{,}15,$$
$$\bar{\mu}^*(x_7, x_{15}) = \lambda_1^* x_7 \otimes \lambda_1^* x_{15} - \lambda_1^* x_{15} \otimes \lambda_1^* x_7$$

So the image of j^* induced by the inclusion j: $\Sigma E \to Q(2)$ is given by

$$Im\ j^* \cong Z_{(2)}\{\Sigma^* x_7, \Sigma^* x_{15}\}.$$

Also the image of δ induced by the collapsing map $Q(2) \to \Sigma Q \wedge \Sigma Q$ is given by

$$Im\ \delta \cong Z_{(2)}\{\delta(v_8 \otimes v_8), \delta(v_8 \otimes v_{16}) = \delta(v_{16} \otimes v_8), \delta(v_{16} \otimes v_{16})\} \oplus S_2$$

where $S_2 \cong Z_{(2)}\{\delta(v_8 \otimes v_{14}), \delta(v_{14} \otimes v_8), \delta(v_{14} \otimes v_{14}), \delta(v_{14} \otimes v_{16}), \delta(v_{16} \otimes v_{14})\}$.

Therefore by (5.4) we obtain the following short exact sequence:

$$0 \to Im\ \delta \hookrightarrow \tilde{H}^*(Q(2)) \xrightarrow{j^*} Z_{(2)}\{\Sigma^* x_7, \Sigma^* x_{19}\} \to 0$$

Thus, denoting by $\bar{v}_4$ and $\bar{v}_8$ the extensions over $Q(2)$ of $\Sigma^* x_7$ and $\Sigma^* x_{15}$, respectively, we obtain the following ring isomorphisms by virtue of [**Th3**]:

$$\begin{aligned} &H^*(Q(2)) \cong Z_{(2)}^{[3]}[\bar{v}_4, \bar{v}_8] \oplus S_2, \\ &\tilde{H}^*(Q(2)) \cdot Im\ \delta = 0, \quad S_2 \subseteq Im\ \delta. \end{aligned} \tag{5.5}$$

We remark that these results are independent of the choice of $\bar{v}_4$ and $\bar{v}_8$.

Similarly one obtains

$$\begin{aligned} &K^*(Q(2)) \cong Z_{(2)}^{[3]}[\bar{w}_4, \bar{w}_8] \oplus S_2^K \\ &\tilde{K}^*(Q(2)) \cdot S_2^K = 0 \\ &\psi^k(\tilde{K}^*(Q(2)) \cdot \tilde{K}^*(Q(2))) \subseteq \tilde{K}^*(Q(2)) \cdot \tilde{K}^*(Q(2)) \\ &Im\ \delta^K \cong Z_(2)\{\delta^K(w_4 \otimes w_4), \delta^K(w_4 \otimes w_8) = \delta^K(w_8 \otimes w_4), \delta^K(w_8 \otimes w_8)\} \oplus S_2^K \\ &S_2^K = Z_{(2)}\{\delta^K(w_4 \otimes w_7), \delta^K(w_7 \otimes w_4), \delta^K(w_7 \otimes w_7), \delta^K(w_7 \otimes w_5), \delta^K(w_8 \otimes w_7)\} \end{aligned} \tag{5.6}$$

where the elements $\bar{w}_4$ and $\bar{w}_8$ are the extensions over $Q(2)$ of $\Sigma^* \xi_7$ and $\Sigma^* \xi_{15}$, respectively.

Furthermore, by (5.3) one obtains

PROPOSITION 5.7.

$$\begin{aligned} \psi^k \delta^K(w_4 \otimes w_7) &\equiv k^{11} \delta^K(w_4 \otimes w_7) + k^4 b(k) \delta^K(w_4 \otimes w_8) \\ \psi^k \delta^K(w_7 \otimes w_4) &\equiv k^{11} \delta^K(w_7 \otimes w_4) + k^9 b(k) \delta^K(w_8 \otimes w_4) \end{aligned}$$

modulo CW filtration > 14.

Now (5.5) and (5.6) imply that $K^*(Q(2))$ and $H^*(Q(2))$ are isomorphic as rings. So we define a ring isomorphism $J : H^*(Q(2)) \to K^*(Q(2))$ by the following

$$\begin{aligned} &J(\bar{v}_i) = \bar{w}_i \quad \text{for } i = 4 \text{ and } 8 \\ &J(\delta(v_{2j} \otimes v_{2j})) = \delta(w_i \otimes w_j) \quad \text{for } i,j = 4,7 \text{ or } 8. \end{aligned} \tag{5.8}$$

By virtue of these relations we introduce Hubbuck operations following [**Hu**]. Then one obtains the following by using (1.5) as in the case $(q,n) = (7,15)$ in §1:

$$\begin{aligned} P^8(\bar{v}_8) &\equiv \bar{v}_8^2 \quad mod\ 2 \\ P^4(\bar{v}_8) &= \alpha \bar{v}_4 \bar{v}_8 \\ P^4(\bar{v}_4) &\equiv \bar{v}_4^2 \quad mod\ 2 \\ P^4(\bar{v}_4) &= \lambda \bar{v}_4^2 + 2\beta \bar{v}_4 \bar{v}_8, \end{aligned} \tag{5.9}$$

where λ, α, $\beta \in Z_{(2)}$ and $\lambda \equiv 1 \ mod\ 2$. (Note that J depends on the choice of $\bar{w}_i$ and hence, so do the exact values of P^i and R^i. But these relations do not depend on the choice of J.)

Next, we will derive a contradiction from the relations of these Hubbuck operations. The relations

$$H^i(Q(2)) = 0 \quad \text{for } i = 10,\ 12,\ 14,\ 18,\ 20,\ 26$$

and Proposition 5.7 imply the following

$$\begin{aligned} R^1(\bar{v}_8) = P^1(\bar{v}_8) = 0, P^1(\bar{v}_4) = R^1(\bar{v}_4) = 0,\\ R^2(\bar{v}_8) = P^2(\bar{v}_8) = 0, P^2(\bar{v}_4) = R^2(\bar{v}_4) = 0,\\ P^3(\bar{v}_4) = R^3(\bar{v}_4) = 0,\\ P^5(\bar{v}_8) = 0, P^5(\bar{v}_4) = 0,\\ P^6(\bar{v}_4) = 0. \end{aligned} \tag{5.10}$$

Further, by (1.4) together with $\nu_2(3^3 - 1) = 1$ (by ignoring the odd multiple) one has

$$2P^3(\bar{v}_8) + 2R^1P^2(\bar{v}_8) + 2^2R^2P^1(\bar{v}_8) + 2^3R^3(\bar{v}_8) \equiv 2^2P^2R^1(\bar{v}_8) + 2^4P^1R^2(\bar{v}_8) \quad mod\ 2^6$$

and hence by (5.10) one obtains the following

$$2P^3(\bar{v}_8) + 2^3R^3(\bar{v}_8) \equiv 0 \quad mod\ 2^6. \tag{5.11}$$

In particular

$$P^3(\bar{v}_8) \equiv 0 \quad mod\ 2^2. \tag{5.11'}$$

Also, (1.4) implies

$$(2^4P^4 + \sum_{i=1}^{4} 2^iR^iP^{4-i})(\bar{v}_4) \equiv 2^2P^3R^1(\bar{v}_4) + 2^4P^2R^2(\bar{v}_4) \quad mod\ 2^6$$

and hence one obtains the following

$$P^4(\bar{v}_4) + R^4(\bar{v}_4) \equiv 0 \quad mod\ 2^2. \tag{5.12}$$

Moreover one obtains

Proposition 5.13.

$$P^6(\bar{v}_8) \equiv 2^3R^6(\bar{v}_8) \quad mod\ 2^4$$

Proof. Equation (1.4) implies

$$2^3P^6(\bar{v}_8) + \sum_{i=1}^{6} 2^iR^iP^{6-i}(\bar{v}_8) \equiv 2^2P^5R^1(\bar{v}_8) + 2^4P^4R^2(\bar{v}_8) + 2^6P^3R^3(\bar{v}_8) \quad mod\ 2^7$$

Recall that $P^4(\bar{v}_8) \in Z_{(2)}\{\bar{v}_4\bar{v}_8\}$, where we have

$$\begin{aligned} R^2(\bar{v}_4\bar{v}_8) &= R^2(\bar{v}_4)\bar{v}_8 + R^1(\bar{v}_4)R^1(\bar{v}_8) + \bar{v}_4 R^2(\bar{v}_8) \\ &= 0 \end{aligned}$$

and hence $R^2P^4(\bar{v}_8) = 0$. So by (5.10) and (5.11) the congruence equation above reduces to

$$2^3P^6(\bar{v}_8) + 2^5R^3R^3(\bar{v}_8) + 2^6R^6(\bar{v}_8) \equiv 2^6P^3R^3(\bar{v}_8) \quad mod\ 2^7$$

where $R^3(\bar{v}_8) \in Z_{(2)}\{\delta(v_8 \otimes v_{14}), \delta(v_{14} \otimes v_8)\}$. Hence by (5.10) we have $R^3R^3(\bar{v}_8) = P^3R^3(\bar{v}_8) = 0$. Thus the congruence equation above reduces to

$$P^6(\bar{v}_8) + 2^3R^6(\bar{v}_8) \equiv 0 \quad mod\ 2^4.$$

QED.

Proposition 5.14.

$$2P^8(\bar{v}_8) \equiv R^4P^4(\bar{v}_8) \quad mod\ 4.$$

Proof. Equation (1.4) implies

$$2P^7(\bar{v}_8) + \sum_{i=1}^{5} 2^iR^iP^{7-i}(\bar{v}_8) \equiv 2^2P^6R^1(\bar{v}_8) + 2^4P^5R^2(\bar{v}_8) \quad mod\ 2^6$$

So by using (5.10), (5.11') and Proposition 5.13 one obtains

$$2P^7(\bar{v}_8) + 2^4R^1R^6(\bar{v}_8) + 2^3R^3P^4(\bar{v}_8) \equiv 0 \quad mod\ 2^6,$$

where $P^4(\bar{v}_8) \in Z_{(2)}\{\bar{v}_4\bar{v}_8 = \delta(v_8 \otimes v_{16})\} \subseteq \tilde{H}^*(Q(2)) \cdot \tilde{H}^*(Q(2))$, and hence

$$R^3P^4(\bar{v}_8) \in Z_{(2)}\{R^3(\bar{v}_4\bar{v}_8)\}$$

By (5.10) and the Cartan formula we have

$$R^3(\bar{v}_4\bar{v}_8) = \bar{v}_4R^3(\bar{v}_8)$$

with $R^3(\bar{v}_8) \in S_2$.

So by (5.5) we have $R^3P^4(\bar{v}_8) = 0$. Therefore we obtain

(5.15) $$2P^7(\bar{v}_8) + 2^4R^1R^6(\bar{v}_8) \equiv 0 \quad mod\ 2^6.$$

Also the equation (1.4) implies

(5.16) $$2^5P^8(\bar{v}_8) + \sum_{i=1}^{5} 2^iR^iP^{8-i}(\bar{v}_8) \equiv 2^2P^7R^1(\bar{v}_8) + 2^4P^6R^2(\bar{v}_8) \quad mod\ 2^6$$

Then by (5.10), (5.11'), Proposition 5.13 and (5.15), one obtains

$$2^5P^8(\bar{v}_8) + 2^4R^1R^1R^6(\bar{v}_8) + 2^5R^2R^6(\bar{v}_8) + 2^4R^4P^4(\bar{v}_8) \equiv 0 \quad mod\ 2^6. \tag{5.17}$$

From (1.4), it follows that

$$2P^1 + 2R^1 \equiv 2^2R^1, \quad mod\ 2^3$$

and hence $P^1 \equiv \pm R^1$ $mod\ 2^2$. Also from (1.4), one has

$$2^3P^2 + 2R^1P^1 + 2^2R^2 \equiv 2^2P^1R^1 + 2^4R^2 \quad mod\ 2^3$$

Then it follows that

$$R^1R^1 = 2R^2 \quad mod\ 2^2$$

Hence

$$R^1R^1R^6(\bar{v}_8) + 2R^2R^6(\bar{v}_8) \equiv 0 \quad mod\ 2^2.$$

Substituting this into (5.17) one obtains

$$2^5P^8(\bar{v}_8) + 2^4R^4P^4(\bar{v}_8) \equiv 0 \quad mod\ 2^6.$$

QED.

PROPOSITION 5.18. *$R^4P^4(\bar{v}_4) \equiv 0 \quad mod\ 4$ or else,*
$\beta \equiv 0 \quad mod\ 2$ where β is as in (5.9).

Proof. Equation (1.4) implies

$$2^5P^8(\bar{v}_4) + \sum_{i=1}^{5} 2^iR^iP^{8-i}(\bar{v}_4) \equiv 2^2P^7R^1(\bar{v}_4) + 2^4P^6R^2(\bar{v}_4) \quad mod\ 2^6$$

So by (1.5) and (5.10) one obtains

$$2^2R^1P^7(\bar{v}_4) + 2^4R^4P^4(\bar{v}_4) \equiv 0 \quad mod\ 2^6 \tag{5.19}$$

Furthermore (1.4) implies

$$2P^7(\bar{v}_4) + \sum_{i=1}^{5} 2^iR^iP^{7-i}(\bar{v}_4) \equiv 2^2P^6R^1(\bar{v}_4) + 2^4P^5R^2(\bar{v}_4) \quad mod\ 2^6$$

So by (5.10) one obtains

$$2P^7(\bar{v}_4) + 2^3R^3P^4(\bar{v}_4) \equiv 0 \quad mod\ 2^6 \tag{5.20}$$

Recall from (5.9) that

$$P^4(\bar{v}_4) = \lambda \bar{v}_4^2 + 2\beta \bar{v}_8$$

So by (5.10) one has

$$R^3 P^4(\bar{v}_4) = 2\beta R^3(\bar{v}_8).$$

Suppose $\beta \not\equiv 0$ *mod* 2. Then by substituting (1.5) into (5.20), one has $P^7(\bar{v}_4) \equiv 0$ *mod* 2^4 and hence

$$2^3 R^3 P^4(\bar{v}_4) \equiv 0 \quad mod\ 2^5, \tag{5.21}$$

so $2^4 \beta R^3(\bar{v}_8) \equiv 0$ *mod* 2^5. Thus

$$R^3(\bar{v}_8) \equiv 0 \quad mod\ 2. \tag{5.22}$$

Then it follows from (5.11) that

$$P^3(\bar{v}_8) \equiv 2^2 R^3(\bar{v}_8) \equiv 0 \quad mod\ 2^3$$

So by rechoosing the ring isomorphism J appropriately (or, in other words, rechoosing the extension $\bar{w}_8 = J(\bar{v}_8)$ appropriately) one obtains the following (due to [**Hu**])

LEMMA 5.23. *One can choose the ring isomorphism J to satisfy $P^3_J(\bar{v}_8) = 0$, if $\beta \not\equiv 0$ mod 2.*

Proof. If $P^3(\bar{v}_8) \neq 0$, we can choose $\bar{v}_{11} \in H^{22}(Q(2))$ so that $P^3(\bar{v}_8) = 2^3 \bar{v}_{11}$. The element $\bar{w}_8' = \bar{w}_8 + \nu \bar{w}_{11}$ with $\nu = \frac{1}{1-2^3}$, where $\bar{w}_{11} = J(\bar{v}_{11})$, is an extension of $\Sigma^* \xi_{15}$. Then from J, we define a new ring isomorphism $J' : H^*(Q(2)) \to K^*(Q(2))$ by setting

$$J'(\bar{v}_8) = \bar{w}_8', \quad J'(\bar{v}_4) = \bar{w}_4$$
$$J'(\delta(v_{2i} \otimes v_{2j})) = \delta^K(w_i \otimes w_j).$$

Then one obtains the following formula modulo higher filtration > 11.

$$\begin{aligned}
\psi^2(J(\bar{v}_8)) &\cong 2^8 J(\bar{v}_8) + 2^8 J(\bar{v}_{11}) \quad mod\ (higher\ filtration > 11) \\
\psi^2(J(\bar{v}_{11})) &\cong 2^{11} J(\bar{v}_{11}) \quad mod\ (higher\ filtration > 11) \\
\psi^2(J'(\bar{v}_8)) &= \psi^2(J(\bar{v}_8) + \nu J(\bar{v}_{11})) \\
&= \psi^2(J(\bar{v}_8)) + \nu \psi^2(J(\bar{v}_{11})) \\
&\cong 2^8 J(\bar{v}_8) + 2^8 J(\bar{v}_{11}) + 2^{11} \nu J(\bar{v}_{11}) \quad mod\ (higher\ filtration > 11) \\
&\cong 2^8 (J(\bar{v}_8) + (2^3 \nu + 1) J(\bar{v}_{11})) \quad mod\ (higher\ filtration > 11) \\
&= 2^8 J'(\bar{v}_8).
\end{aligned}$$

Thus $P^3_{J'}(\bar{v}_8) = 0$ (Note that the operation $P^3_{J'}$ with respect to J' is different from P^3 $= P^3_J$ with respect to J). The operations $P^i_{J'}$ and $R^i_{J'}$ satisfy all the formulae given above for the ones with respect to the general 'J'. So, we may consider the ring isomorphism J to satisfy $P_J^3 = 0$. QED.

Hence from (5.11), (5.21) and (5.20), it follows that

$$\begin{aligned} R^3(\bar{v}_8) &\equiv 0 \quad mod\ 2^3, \\ R^3P^4(\bar{v}_4) &\equiv 0 \quad mod\ 2^4, \\ 2P^7(\bar{v}_4) &\equiv 0 \quad mod\ 2^6. \end{aligned}$$

Substituting them into (5.19) one obtains

$$2^4R^4P^4(\bar{v}_4) \equiv 0 \quad mod\ 2^6$$

That is, if $\beta \not\equiv 0$ mod 2, then $R^4P^4(\bar{v}_4) \equiv 0$ mod 4. QED.

Now these two propositions, Proposition 5.14 and 5.18, give us a contradiction.

By Proposition 5.14, we have the following equation mod 4.

$$\begin{aligned} 0 \not\equiv 2\bar{v}_8^2 &\equiv R^4P^4(\bar{v}_8) \\ &\equiv R^4(\alpha\bar{v}_4\bar{v}_8) \\ &\equiv \alpha R^4(\bar{v}_4)\bar{v}_8 + \alpha\bar{v}_4R^4(\bar{v}_8) \end{aligned} \tag{5.24}$$

by (5.10) and the Cartan formula, where $R^4(\bar{v}_8) \in Im\ \delta$ and hence $\bar{v}_4R^4(\bar{v}_8) = 0$ by (5.5). Furthermore, using (5.10), one obtains the following from (1.4):

$$2^4P^4(\bar{v}_4) + 2^4R^4(\bar{v}_4) \equiv 0 \quad mod\ 2^6$$

which implies

$$R^4(\bar{v}_4) \equiv -P^4(\bar{v}_4) \equiv -\lambda\bar{v}_4^2 - 2\beta\bar{v}_8 \quad mod\ 4. \tag{5.25}$$

Hence from (5.24), it follows that

$$0 \not\equiv 2\bar{v}_8^2 \equiv -2\alpha\beta\bar{v}_8^2 \quad mod\ 4.$$

Then it follows that

$$\alpha\beta \equiv 1 \quad mod\ 2; \quad \text{in particular,} \quad \beta \equiv 1 \quad mod\ 2. \tag{5.26}$$

Since $\beta \not\equiv 0$ mod 2, Proposition 5.18 implies

$$\begin{aligned} 0 \equiv R^4P^4(\bar{v}_4) &\equiv R^4(\lambda\bar{v}_4^2 + 2\beta\bar{v}_8) \\ &\equiv 2\lambda\bar{v}_4R^4(\bar{v}_4) + 2\beta R^4(\bar{v}_8) \end{aligned} \tag{5.27}$$

by (5.10) and the Cartan formula.

Here, by (5.25), we have

$$2\lambda\bar{v}_4R^4(\bar{v}_4) \equiv 0 \quad mod\ 4.$$

Also by (1.4) using (5.10) and Lemma 5.23 we have

$$2^4P^4(\bar{v}_8) + 2^4R^4(\bar{v}_8) \equiv 0 \quad mod\ 2^6$$

and hence

$$R^4(\bar{v}_8) \equiv -P^4(\bar{v}_8) = -\alpha\bar{v}_4\bar{v}_8 \quad mod\ 4.$$

Substituting them into (5.27) we obtain

$$0 \equiv R^4P^4(\bar{v}_4) \equiv -2\alpha\beta\bar{v}_4\bar{v}_8,$$

which contradicts (5.26).

Thus we have shown that there exists no Poincaré complex with GW space structure whose cohomology ring is an exterior algebra of type (7,15). QED.

§6. Proof of the main theorem

Let E be a Poincaré complex of type (q,n). One may assume that E has a cell structure $S^q \cup_\alpha e^n \cup e^{n+q}$ with $\alpha \in \pi_{n-1}(S^q)$.

[The case $n = q$.] Then E has a cell structure $S^q \vee S^q \cup e^{2q}$. We define an inclusion ι : $Q \hookrightarrow E$ by the canonical inclusion $S^q \vee S^q \subset E$. Since Q is desuspendable, there is an axial map $\mu : Q \times Q \to E$ with axis ι by the assumption. Denote by $Q(2)$ the cofibre of the Hopf construction $H(\mu) : Q * Q \to \Sigma E$ of the map μ. Then one has

$$H^*(Q(2); Z/2) \cong Z/2^{[3]}[v_{q+1}, vn+1]$$

and $Sq^1QH^*(Q(2); Z/2) = 0$. Then by Proposition 1.7 one has $\{q, n\} \subseteq \{1, 3, 7\}$.

[The case $n = q+1$.] Then E has a cell structure $S^q \cup_{m\iota} e^{q+1} \cup e^{2q+1}$ where $m\iota \in \pi_q(S^q)$ $\cong Z$. If m is odd, then $E \simeq_2 S^{2q+1}$. So inheriting a GW space structure from E, $S^{2q+1}_{(2)}$ becomes a GW space and, into particular, S^{2q+1} becomes a Hopf space, whence $q = 1$ or 3. Therefore (q,n) = (1,2) or (3,4) and $E \simeq_2 S^3$ or S^7. If m is even and $q = 3$, then $H^*(E; Z/2) \cong \wedge(x_q, x_n)$. Putting $Q = S^q \cup_{m\iota_q} e^{q+1}$, we get a space $Q(2)$ as in case when $n \leq q$. Then one has

$$H^*(Q(2); Z/2) \cong Z/2^{[3]}[v_{q+1}, v_{q+2}]$$

Then Proposition 1.7 says that $q = 1$, which is a contradiction. Hence (q,n) = (1,2) or (3,4), and $E \simeq_2 S^7$ if $q = 3$.

[The case $q + 1 < n < 2q$.] Then E has a cell structure $S^q \cup_\alpha e^n \cup e^{n+q}$ with $\alpha \in \pi_{n-1}(S^q)$. By assumption, $n < 2q$ and α is a suspended element, that is, $Q = S^q \cup_\alpha \cup e^n$ is desuspendable. There is a map $\mu : Q \times Q \to E$ since E is a GW space. Quite similarly to the above cases, one can construct a space $Q(2)$ satisfying

$$H^*(Q(2); Z/2) = Z/2^{[3]}[v_{q+1}, v_{n+1}]$$

From Proposition 1.7 it follows that (q,n) = (3,5)

[The case $n = 2q > 2$.] Then E has a cell structure $S^q \cup_\alpha e^n \cup e^{n+q}$ with $\alpha \in \pi_{n-1}(S^q)$. If q is odd, one has $H^*(E; Z) \cong \wedge(x_q, x_n)$ which contradicts Proposition 3.3. Hence q is even. Take the bottom inclusion $j : S^q \to E$. Then the map $j \circ [\iota, \iota] : S^{2q-1} \to E$ is null homotopic, since E is a GW space. For dimensional reasons there is a map $\lambda : S^{2q-1} \to S^{2q-1}$ such that the following diagram homotopy commutes:

$$\begin{array}{ccc} S^{2q-1} & \xrightarrow{[\iota_q,\iota_q]} & S^q \\ \lambda\downarrow & & \iota_q\downarrow \\ S^{2q-1} & \xrightarrow{\alpha} & S^q \end{array}$$

Thus $[\iota_q, \iota_q] = \lambda\alpha$, where $[\iota_q, \iota_q]$ is an element in the free part of $\pi_{2q-1}(S^q)$ and so is α. Therefore we get

$$\begin{aligned} H^*(E;Z) &\cong Z[x_q]/(x_q^4) && \text{if } \lambda = 2 \\ &\cong Z\{x_q, x_{2q}, x_q x_{2q}\},\ x_q^2 = 2x_{2q} && \text{if } \lambda = 1. \end{aligned}$$

So it follows from Proposition 2.1 that $(q,n) = (2,4)$. That is,

$$H^*(E;Z) \cong H^*(CP^3;Z).$$

[The case $2q < n$.] Then E has the the homotopy type of $S^q \cup_\alpha e^n \cup e^{q+n}$ with $\alpha \in \pi_{n-1}(S^q)$. By Proposition 3.3, one has

$$H^*(E;Z) \cong \wedge(x_q, x_n)$$

Hence from Theorem 4.2 and Proposition 5.1 and 5.2, it follows that $(q,n) = (1,3)$, $(1,7)$ or $(3,7)$.

Remark. *When $(q,n) = (3,7)$, the attaching element α of the 7-cell in E is of the form $\alpha = \lambda\omega$ with λ odd or $\lambda \equiv 0$ mod 4, where ω is the Blakers-Massey element in $\pi_6(S^3)$.*

In fact, if λ is odd or $\lambda \equiv 0$ mod 4, the pullback by λ from the principal bundle $Sp(2) \to S^7$ is known to be a Hopf space and so it is a GW space. If $\lambda \equiv 2$ *mod* 4, α is desuspendable at 2 and so is the space $Q = (S^3 \cup_\alpha e^7)_{(2)}$. Then one can construct a space $Q(2)$ from which one can deduce a contradiction to the result of Sigrist-Suter [**S-S**] (since the result in [**S-S**] is essentially a result localised at $p = 2$).

We would like to propose the following

Conjecture. *If E is a 1-connected finite GW space such that $H^*(E;Z)$ is an exterior algebra on odd degree generators, then E is a Hopf space.*

Appendix

Let E and B be connected CW complexes and consider a fibration

(A.1) $$F \overset{\iota}{\hookrightarrow} E \overset{\pi}{\to} B$$

with fibre F a (not necessarily connected) CW complex. It gives rise to the following two fibrations:

(A.2) $$\Omega B \overset{q}{\to} F \overset{\iota}{\to} E,$$

(A.3) $$\Omega E \overset{\Omega\pi}{\to} \Omega B \overset{q}{\to} F.$$

Now suppose that ι is null homotopic. It follows from (A.2) that q has a right inverse $s : F \to \Omega B$. So the homotopy exact sequence of (A.3) splits and we obtain

$$\pi_*(\Omega B) \cong \pi_*(\Omega E) \oplus \pi_*(F),$$

where the above isomorphism is induced by the map $h = \mu \circ (\Omega\pi \times s) : \Omega E \times F \to \Omega B$ with μ the loop multiplication of ΩB. Thus h is a homotopy equivalence, since ΩB and ΩE have the homotopy type of a CW complex. Hence we obtain

(A.4) $$h : \Omega E \times F \simeq \Omega B$$

Thus the following hold for any space W:

(A.5) $$\begin{array}{ll} 1 \to [W, \Omega E] \overset{\Omega\pi_*}{\hookrightarrow} [W, \Omega B] & \text{as groups,} \\ [W, \Omega B] \cong [W, \Omega E] \times [W, F] & \text{as sets.} \end{array}$$

Here we would like to introduce a notion of GW action. A GW action of E along $\pi : E \to B$ is a map

(A.6) $$\nu : \Sigma\Omega E \times \Sigma\Omega B \to B$$

with axes $\Sigma\Omega E \to E \overset{\pi}{\to} B$ and $\Sigma\Omega B \to B$, where a map $\Sigma\Omega X \to X$ is the evaluating map.
Then we have

Theorem A.7. *If ι is null-homotopic in (A.1) and if B admits a GW action of E along π (see (A.6)), then the following three statements hold:*
(i) E is a GW space and F is an H-space.
(ii) If B is a GW space, then F is a homotopy abelian H-space.
(iii) B is a GW space if and only if the Sameleson product $< s,s >$ is trivial for a right inverse s of q.
(iv) If there is an H-map s which is a right inverse of q and if F is homotopy abelian, then B is a GW space and (A.4) is an H-equivalence.

Proof. (i) By [**O**, Theorem 2.7], the image of $\Omega\pi_*$ of (A.5) is contained in the center of $[W,\Omega G] \cong [\Sigma W,G]$ for any W, since a map from a suspension space to a space X can be decomposed through the evaluating map $\Sigma\Omega X \to X$. Furthermore $\Omega\pi_*$ is a monomorphism by (A.5), and hence $[W,\Omega G]$ is an abelian group for any W, which implies that E is a GW space. Since F is a retract of a loop space ΩB, it is an H-space.
(ii) Let us define the multiplication $\bar\mu$ of F by putting $\bar\mu = q \circ \mu \circ (s \times s)$, where we denote by μ the loop multiplication of ΩB. As μ is homotopy abelian, so is $\bar\mu$.
(iii) First suppose that B is a GW space. Since ΣF is a suspension space, the Whitehead product $[ad(s),ad(s)]$ is trivial for the adjoint map $ad(s) : \Sigma F \to B$ of s. Recall that $[ad(s),ad(s)] = \pm ad < s, s >$, where $ad < s, s >$ denotes the adjoint of the Samelson product of s. Thus we obtain $ad < s, s > = *$.

Conversely suppose that $ad < s, s > = *$. For simplicity we write $\mu(x, y) = x \cdot y$. Then by the homotopy associativity of μ, we obtain the following homotopy.

$$\begin{aligned} h(x, y) \cdot h(\bar{x}, \bar{y}) &= (\Omega\pi(x) \cdot s(y)) \cdot (\Omega\pi(\bar{x}) \cdot s(\bar{y})) \\ &\simeq (\Omega\pi(x) \cdot (s(y) \cdot \Omega\pi(\bar{x}))) \cdot s(\bar{y}) \end{aligned}$$

The image of $\Omega\pi_*$ is contained in the center as is seen in (i), and so we obtain

$$s(y) \cdot \Omega\pi(\bar{x}) \simeq \Omega\pi(\bar{x}) \cdot s(y).$$

Then from the homotopy commutativity, it follows that

$$\begin{aligned} h(x, y) \cdot h(\bar{x}, \bar{y}) &\simeq (\Omega\pi(x) \cdot (\Omega\pi(\bar{x}) \cdot s(y))) \cdot s(\bar{y}) \\ &\simeq (\Omega\pi(x) \cdot \Omega\pi(\bar{x})) \cdot (s(y) \cdot s(\bar{y})). \end{aligned} \tag{A.8}$$

Recalling that the loop map $\Omega\pi$ is an H-map, one has

$$\Omega\pi(x) \cdot \Omega\pi(\bar{x}) \simeq \Omega\pi(x \cdot \bar{x})$$

where we use the same symbol $\cdot$ to denote the loop multiplications of ΩB and ΩE. Let us recall that ΩE is homotopy abelian by (i), so that

$$\Omega\pi(x \cdot \bar{x}) \simeq \Omega\pi(\bar{x} \cdot x).$$

Thus we obtain

$$\Omega\pi(x) \cdot \Omega\pi(\bar{x}) \simeq \Omega\pi(\bar{x}) \cdot \Omega\pi(x).$$

From the hypothesis $< s,s > = *$, it follows that $s(y) \cdot s(\bar{y}) \cdot s(y)^{-1} \cdot s(\bar{y})^{-1} \simeq *$. Hence it follows that

$$s(y) \cdot s(\bar{y}) \simeq s(\bar{y}) \cdot s(y).$$

Summing up we get

$$\begin{aligned} h(x, y) \cdot h(\bar{x}, \bar{y}) &\simeq (\Omega\pi(x) \cdot \Omega\pi(\bar{x})) \cdot (s(y) \cdot s(\bar{y})) \\ &\simeq h(\bar{x}, \bar{y}) \cdot h(x, y), \end{aligned}$$

that is,

$$\mu \circ (h \times h) \simeq \mu \circ T \circ (h \times h).$$

Since h is a homotopy equivalence in (A.4), it then follows that

$$\mu \simeq \mu \circ T,$$

that is, ΩB is homotopy abelian. Thus B is a GW space.

(iv) Let $s : F \to \Omega B$ be an H-map which is a right inverse of q. Then the H-deviation $HD(s)$ of s satisfies $HD(s) \simeq *$, where the H-deviation $HD(s) : F \wedge F \to \Omega B$ is given by

$$HD(s)(x \wedge y) = s(x) \cdot s(y) \cdot s(x + y)^{-1}$$

where + denotes the multiplication of F. It follows that

$$HD(s)(y\wedge x) = s(y)\cdot s(x)\cdot s(y+x)^{-1}$$

Since F is homotopy abelian, we have $s(x+y) \simeq s(y+x)$. Thus we have

$$\begin{aligned} HD(s)(x\wedge y)\cdot HD(s)(y\wedge x)^{-1} &\simeq s(x)\cdot s(y)\cdot s(x+y)^{-1}\cdot s(y+x)\cdot s(x)^{-1}\cdot s(y)^{-1} \\ &\simeq s(x)\cdot s(y)\cdot s(x)^{-1}\cdot s(y)^{-1} \\ &= < s, s >(x\wedge y). \end{aligned}$$

This implies that $< s,s > \simeq *$, and hence B is a GW space by (iii). Further, by (A.8) we have

$$\begin{aligned} \mu \circ (h\times h)((x,y),(\bar{x},\bar{y}) &\simeq h(x,y)\cdot h(\bar{x},\bar{y}) \\ &\simeq (\Omega\pi(x)\cdot\Omega\pi(\bar{x}))\cdot(s(y)\cdot s(\bar{y})) \end{aligned}$$

which by using the H-structure of maps s and $\Omega\pi$, changes up to homotopy as follows:

$$\begin{aligned} &\simeq \Omega\pi(x\cdot\bar{x})\cdot s(y+\bar{y}) \\ &= h(x\cdot\bar{x}, y+\bar{y}). \end{aligned}$$

This implies that h is an H-map and hence ΩB is H-equivalent to $\Omega E \times F$. QED.

COROLLARY A.9. *(i) The standard lens space* $L(m) = S^3/(Z/mZ)$ *is a GW space for all* $m \geq 1$.

(ii) $CP^3 = S^7/T^1$ *is a GW space.*

Proof. (i) Put $F = Z/mZ$, $E = S^3$ and $B = L(m)$. They satisfy the conditions of Theorem A.7. So it suffices to show that $s : F \to \Omega E$ is an H-map. The H-deviation of s is in the set $[F\wedge F,\Omega E] \cong [F*F,E \cong [\vee_\alpha S^1_\alpha,S^3] \cong \oplus_\alpha\pi_1(S^3) = 0$. Hence $HD(s) \simeq *$, that is, s is an H-map. From (iv) of Theorem A.7, it follows that $B = L(m)$ is a GW space.

(ii) Put $F = T^1$, $E = S^7$ and $B = CP^3$. They satisfy the conditions of Theorem A.7, since CP^3 is a Whitehead space and $\Sigma\Omega CP^3$ has the homotopy type of a wedge sum of spheres. The H-deviation of $s : F \to \Omega G$ is in the set $[F\wedge F,\Omega E] \cong \pi_3(S^7) = 0$, whence s is an H-map. From (iv) of Theorem A.7, it follows that $B = CP^3$ is a GW space. QED.

REMARK. *If we put* $F = T^1$, $E = S^3$ *and* $B = S^2$, *they also satisfy the conditions of Theorem A.7, but a splitting* $s : F \to \Omega B$ *cannot be an H-map. In fact, its H-deviation is the adjoint of the Hopf map* $\eta : S^3 \to S^2$, *and* S^2 *is not* a GW space.

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