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H-spaces with generating subspaces*

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Synopsis

For an H-space with a generating subspace, we construct a space whose K-cohomology is a direct sum of a truncated polynomial algebra and an ideal, which enables technical restrictions to be removed from several known results in the homotopy theory of H-spaces.

0. Introduction

We consider an H-space in the category of connected finite CW-complexes with base point and mappings preserving base points (or in the p-localised category of such spaces and mappings at a prime p). On that category, the coefficient ring of (co)homology theories is the ring of integers (or localised integers, respectively) which we denote by R. We call a subspace Q of a space X a generating subspace if the inclusion $j: Q \to X$ induces an isomorphism $j^!: QK^*(X; R) \to \tilde{K}^*(Q; R)$, where we denote by $QK^*(-; R)$ the indecomposable quotients and by $\tilde{K}^*(-; R)$ the augmentation ideal of the $\mathbb{Z}/2\mathbb{Z}$-graded complex K-cohomology $K^*(-; R)$; that is, the generators of $K^*(X; R)$ are represented by Q. Classical Lie groups $U(n)$, $SU(n)$ and $SP(n)$ have such generating subspaces [**10**].

Let us consider an A_m-space X for $m \geqq 2$ (see [**15**]). Then there exist projective spaces $P(k)$, $k \leqq m$ with $P(m) \supset P(m-1) \supset \ldots \supset P(1) = \Sigma X$ where Σ is the suspension functor. If X is A_k-primitive (see [**8**]), then $K^*(P(k); R)$ has the form $M(k) \oplus S_k$, where $M(k)$ is a polynomial algebra truncated at height $k+1$ and S_k is a free R-module and an ideal (see [**7**]). In addition $\psi^l(S_k) \subset S_k$ for all Adams operations ψ^l. However, it is not known if every A_m-space supports an A_m-primitive A_m-structure, although it is automatically A_{m-1}-primitive. We construct a space $Q(m)$ by expanding $P(m-1)$ when X has a generating subspace Q and show by refining the arguments in [**1**] or [**7**], that $K^*(Q(m); R)$ has the form $M(m) \oplus \hat{S}_m$, which enables us to compute Adams operations ψ^k without assuming the A_m-primitivity.

From now, we often abbreviate the coefficient ring of (co)homology theories which we always assume to be the ring R. If X is a 1-connected finite H-space, then $K^*(X)$ has no torsion by [**11**] and [**12**]. Since Chern character filtration makes $K^*(X)$ a torsion free graded Hopf algebra, $K^*(X)$ is an exterior algebra on odd dimensional generators.

Our main theorem is stated as follows.

THEOREM 0.1. *Let X be a connected A_m-space with a generating subspace Q, $m \geqq 2$. Then there exists a space $Q(m)$ with $P(m) \supset Q(m) \supset P(m-1) \supset \Sigma X \supset$*

* Dedicated to Professor Shôrô Araki on his sixtieth birthday.

ΣQ. *If X is* 1*-connected or, more generally, has no torsion in its K-cohomology, then there is an isomorphism of R-algebras*

$$K^*(Q(m)) \cong M \oplus \hat{S}_m, \quad \textit{where}$$

(1) *the restriction of the homomorphism $\tilde{K}^*(Q(m)) \to \tilde{K}^*(\Sigma Q)$ to M is surjective, and for any choice of pull-backs $\{u_i, 1 \leqq i \leqq r\}$ of additive generators of $\tilde{K}^*(\Sigma Q)$ where r is the rank of X,*

(2) *$M \cong R^{[m+1]}[u_1, \ldots, u_r]$ is a polynomial algebra truncated at height $m+1$,*

(3) *$\psi^k(\bar{M} \cdot \bar{M}) \subseteq \bar{M} \cdot \bar{M} \subset \bar{M}$ for all k, where $\bar{M}$ is the augmentation ideal of M,*

(4) *$\hat{S}_m$ is an ideal and a free R-module with $\hat{S}_m \cdot \tilde{K}^*(Q(m)) = 0$,*

(5) *the restriction of the homomorphism $K^*(Q(m)) \to K^*(P(m-1))$ to $\hat{S}_m$ is injective with image in S_{m-1}.*

The author does not know the ψ^k-invariance of $\hat{S}_m$ unless assuming the A_m-primitivity, in general. But in some cases, M can be chosen to be closed under the action of ψ^k for all k.

If, in particular, the restriction of M to $P(m-1)$ is closed under the action of ψ^k, so is M. More generally, we obtain:

COROLLARY 0.2. *If, in addition, there exists a submodule $L \subseteq K^*(P(m-1))$ with $L \subseteq QK^*(P(m-1))$ and $\psi^k(L) \subseteq L$ for all k and the restriction of the homomorphism $\tilde{K}^*(P(m-1)) \to \tilde{K}^*(P(m-1))/S_{m-1}$ to L is injective, then there is a submodule $L' \subset QM$ with $L'|_{P(m-1)} = L$ and $\psi^k(L') \subset L' + \bar{M} \cdot \bar{M}$. In particular, if further $L \cong \tilde{K}^*(\Sigma Q)$ for $m \geqq 2$, then we may assume that $\psi^k(M) \subseteq M$ for all k.*

In the p-localised category, we have another sufficiency condition obtained by Corollary 0.2.

COROLLARY 0.3. *In the case $m = p$ a prime, if $K^*(X)$ has a spherical generator $x = f^!(y)$, $y \in K^*(S^{2n-1}_{(p)})$ where $f: X \to S^{2n-1}_{(p)}$ is an A_{p-1} mapping, then we can choose a generator u in M such that $\psi^k(u) - k^n u$ is in $\bar{M} \cdot \bar{M}$ for all k.*

This corollary is related to the results of **[19]**, in the case $p = 2$. But in the case $p \geqq 3$, we need the additional hypothesis that f is an A_{p-1}-mapping.

The spaces $Q(2)$ and $Q(3)$ for pull-backs of $Sp(2)$ were introduced in **[13]** and **[14]** to show the non-existence of A_m-structures ($m = 2, 3$) on the pull-backs of $Sp(2)$. To show the ring structure of $K^*(Q(m))$, A_m-primitivity is established by dimensional arguments in **[13]** and **[14]**.

We would like to bring to the reader's attention an H-space without torsion in its (ordinary) homology with coefficient in R. If $H_*(X)$ is torsion free, then $H^*(X)$ is also an exterior algebra on odd dimensional generators. Then by the arguments of the proof of Theorem 0.1, we obtain:

THEOREM 0.4. *If, further, X has no torsion in its homology, then there is an isomorphism of R-algebras $H^*(Q(m)) \cong N \oplus \hat{T}_m$, where $N = R^{[m+1]}[v_1, \ldots, v_r]$ and $\hat{T}_m$ is an ideal and a free R-module.*

These results enable us to remove technical restrictions from several known results in the homotopy theory of H-spaces.

It is known that the first non-vanishing homotopy group of a 2-torsion free finite H-space must occur in dimension 1, 3 or 7. In our case, this can also be proved using the same argument as that given in [**17**]. Moreover from [**18**], we obtain the following:

PROPOSITION 0.5. *Let X be a 2-torsion free finite H-space with a generating subspace Q. If there is a submodule $L \subseteq H^*(\Sigma X; \mathbb{F}_2)$ with $L \cong H^*(Q; \mathbb{F}_2)$ and $Sq^i L \subset L$ for all Steenrod square operations Sq^i, then the first non-vanishing homotopy group occurs in dimension* 1, 3 *or* 7. *Furthermore, the action of Steenrod squares satisfies*

$$Sq^{2j}(QH^{2i-1}(X; \mathbb{F}_2)) = QH^{2i+2j-1}(X; \mathbb{F}_2)$$

and

$$Sq^{2j}(QH^{2i+2j-1}(X; \mathbb{F}_2)) = 0,$$

if $\binom{2i-1}{2j}$ *is odd.*

From [**3**, Theorem 1.1(b)], we remove the condition "the space is (mod 2) standard". This, together with [**3**, Theorem 1.1(2)], implies:

COROLLARY 0.6. *The space of a Stiefel manifold which supports a* (mod 2) *H-space is that of a Lie group or S^7.*

From a theorem from [**5**], we remove the condition "X is A_p-primitive" and obtain:

PROPOSITION 0.7. *Let $X = S^{n_1} \times \ldots \times S^{n_r}$ be a 1-connected* mod p *A_p-space for an odd prime p. Then*

(i) *for each i, $m_i \in \{3, 5, \ldots, 2p-1\}$*

and

(ii) $\operatorname{rank}_R H^3(X) \geqq \operatorname{rank}_R H^{2p-1}(X)$.

Hence, from the results of [**5**] and [**2**], we can remove the condition "X is A_3-primitive" and obtain the following corollary to Proposition 0.7.

COROLLARY 0.8. *Let $X = S^{n_1} \times \ldots \times S^{n_r}$ be a connected* mod 3 *A_3-space. Then X has the* mod 3 *homotopy type of a product of Lie groups $U(1)$'s , $SU(2)$'s and $SU(3)$'s.*

Before we state our last corollary, we mention that the condition "$j^!: QK^*(X) \to \tilde{K}^*(Q)$ is isomorphic" can be weakened slightly.

THEOREM 0.9. *If we assume that $j^!: Q\tilde{K}^*(X) \to K^*(Q)$ is surjective, then we obtain another complex $Q(m, j)$ with similar properties to those of $Q(m)$, except that M and $\hat{S}_m$ must be replaced by $M(m, j) = R^{[m+1]}[u_1, \ldots, u_r]/R^m(j^!)$ and $\hat{S}_m(j)$, respectively, where $R^m(j^!)$ is the ideal generated by all products of m elements u_i's whose restrictions to ΣX are in the kernel of $\Sigma j^!$. If the space has no torsion, a similar result holds for the ordinary cohomology.*

Using this together with the proof of Corollary 0.2, we can remove the condition "X is A_3-primitive" from [**4**, Theorem 1.1] and obtain:

COROLLARY 0.10. *Let the integral homology of X have no 2-torsion. Then $S^7 \times X$ does not support* (mod 2) *an A_3-structure.*

1. The construction of $Q(m)$

Let X be an A_m-space with generating subspace $j: Q \to X$. By the definition of an A_m-structure in **[15]**, there exists a sequence of quasi-fibrations $p_k: E^k(X) \to P(k-1)$ for $k \leqq m$ with fibre X, where $P^k(X)$ is called the projective k-space, with the following properties:

$$\left.\begin{aligned} &E^k \simeq X * \ldots * X \text{ (homotopic to } k\text{-fold join),} \\ &P(k) = P(k-1) \cup_{p_k} D^k, \end{aligned}\right\} \tag{1.1}$$

where $E^k \subseteq D^k$, $D^{k-1} \subseteq E^k$ and D^k is contractible for $k \leqq m$.

To use the results of **[16]**, both p_k and the homotopy equivalence above have to be triad mappings. By the proofs of [**15**, Theorems 11 and 12], there are homotopy equivalences:

$$\lambda_k: \hat{E}^k \equiv E^{k-1} \cup_{\mu_{k-1}} X \times CE^{k-1} \xrightarrow{\simeq} E^k,$$
$$\lambda'_k: \hat{P}(k-1) \equiv P(k-2) \cup_{\nu_{k-1}} CE^{k-1} \xrightarrow{\simeq} P(k-1),$$

where $\nu_{k-1}: E^{k-1} \to P(k-2)$ is obtained by ignoring the first factor from μ_{k-1} and $\mu_{k-1}: X \times E^{k-1} \to E^{k-1}$ satisfies the following conditions:

$$\left.\begin{aligned} &\mu_{k-1}|_{X \times \{*\}} \sim id_X \text{ for } k = 2 \text{ and } * \text{ for } k \geqq 3, \\ &\mu_{k-1}|_{E^{k-1}} \sim id_{E^{k-1}}, \\ &\text{and therefore} \\ &\nu_{k-1} \sim p_{k-1}, \end{aligned}\right\} \tag{1.2}$$

where " $\sim$ " means "is homotopic to". By using these homotopy equivalences, p_k can be regarded as a triad mapping as in **[7]**. We define $\pi_k: \hat{E}^k \to \hat{P}(k-1)$ by setting $\pi_k|_{E^{k-1}} = p_{k-1}$ and $\pi_k|_{X \times CE^{k-1}} = pr_{CE^{k-1}}$, the projection to the factor CE^{k-1}. Then π_k can be regarded as p_k up to homotopy and is a triad mapping with respect to the standard triad decomposition of CE^{k-1} given in **[7]**. Also we note that $X * Y = CX \times Y \cup X \times CY$ and $CX \times Y \cap X \times CY = X \times Y$. Then by the proof of [**15**, Theorem 11], we obtain:

PROPOSITION 1.1. *There exists a series of homotopy equivalences $h_k: X * \ldots * X \to \hat{E}^k$ such that*

(i) $h_1 = id$,

(ii) $h_{k+1}|_{X \times \{*\}} = id$, $h_{k+1}|_{\{*\} \times (X * \ldots * X)} = \lambda_k \circ h_k$

and

(iii) $h_{k+1}|_{X \times (X * \ldots * X)}(x, e) = (x, \hat{\mu}_k(x, h_k(e)))$,

where $\hat{\mu}_k: X \times E^k \to E^k$ gives the inverse action of $\mu_k: X \times E^k \to E^k$.

Remark 1.2. No proof is given for [**15**, Corollary 26]; indeed, the result is not correct for $j = 1$ or i. Moreover $X \times \mathscr{D}^i$ cannot map into $\mathscr{E}_i$ in the way described in **[15]** except for the case when X is a monoid. But one can avoid these difficulties **[6]**. By changing faces of the Stasheff complex, we obtain [**15**, Corollary 26] for $2 \leqq j \leqq i-1$ (the other cases are not needed). We define μ directly by using [**15**, Corollary 26] and one can show the homotopy equivalence of $(\mathscr{E}_i, \mathscr{E}_{i-1})$ and

$((\mathscr{E}_{i-1} \cup_\mu X \times C\mathscr{E}_{i-1}), \mathscr{E}_{i-1})$. We remark further that $X \times \mathscr{E}_{i-1}$ is not included in $\mathscr{E}_i$ by the above homotopy equivalence. The details will appear in **[9]**.

Let us define $Q(m)$ as the homotopy cofibre of the following mapping:

$$\hat{\pi}_m = \lambda'_m \circ \pi_m \circ h_m \circ (j * \ldots * j): Q * \ldots * Q \to P(m-1), \tag{1.3}$$

where λ'_m and h_m are the homotopy equivalence. Then it follows that $Q(m) \supset P(m-1) \supset P(1) = \Sigma X \supset \Sigma Q$, where we denote by Σ the suspension functor. We may regard $P(m)$ as the mapping cone of $\lambda'_m \circ \pi_m \circ h_m$ which includes $Q(m)$.

2. Proof of Theorem 0.1

We establish the algebra structure of $K^*(Q(m))$. Firstly, we mention that $K^*(X)$ is an exterior algebra on odd dimensional elements. Let P be the module generated by representatives of $QK^*(X) \cong \tilde{K}^*(Q)$ for $m = 2$ or the module of primitives for $m \geqq 3$. We choose and fix the R-module basis of P as $\{x_i; 1 \leqq i \leqq r\}$.

Let us recall the ring structure of $K^*(P(m-1))$. By a corollary of **[7]**, it follows that X is A_{m-1}-primitive, that is, there are elements u_i in $\tilde{K}^*(P(m-1))$ such that $u_i|_{\Sigma X} = s^!(x_i)$, where we denote by $s^!$ the suspension isomorphism. In addition, by **[7]**, Theorem A], there is the following isomorphism of algebras:

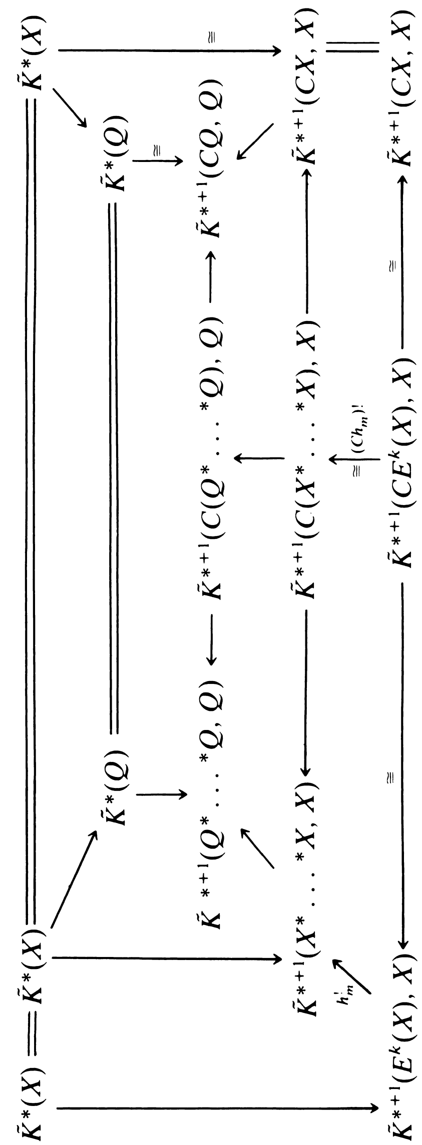
$$\left.\begin{aligned} &K^*(P(k-1)) \cong R^{[k]}[u_1, \ldots, u_r] \oplus S_{k-1}, \\ \text{with}\quad &S_k = \delta_k(\tilde{S}_k), \\ &\tilde{S}_k = \sum_{i \geqq 1} \tilde{K}^*(X) \otimes \ldots \otimes D \otimes \ldots \otimes \tilde{K}^*(X), \quad k \leqq m, \end{aligned}\right\} \tag{2.1}$$

where δ_k is the Mayer–Vietoris coboundary for $P(k)$ for $k \leqq m$, D is the module of decomposables of $K^*(X)$ and we regard $\tilde{K}^*(E^k)$ as $\tilde{K}^*(X) \otimes \ldots \otimes \tilde{K}^*(X)$ by the homotopy equivalence (1.1).

Again by a corollary of **[7]**, the obstruction to be A_m-primitive lies in S_m of the E_2-term. So, in the E_1-term, the obstruction $\pi^!_m(u_i)$ lies in $\tilde{S}_m$ and depends on the choice of pullback u_i. Let us recall the definition of $Q(m)$ (1.3). Since $\tilde{S}_m = \operatorname{Ker}(j * \ldots * j)^!$, we find that $\hat{\pi}^!_m(u_i) = 0$. Hence every u_i is extendable to $Q(m)$. We fix a system of extensions to $K^*(Q(m))$ of u_i and denote it by the same symbol u_i.

Proposition 2.1. *Each $j^!(x_i)$ is transgressive with respect to $\hat{\pi}_m$ in the sense of* **[16]** *and its transgression image is u_i.*

Proof. The inclusion mapping j and the homotopy equivalence h_m induce the following commutative diagram (see p. 6). Here vertical lines are connecting homomorphisms for suitable pairs and all other lines except for $h^!_m$ and $(Ch_m)^!$ are induced by inclusions. Firstly, we mention that $\hat{\pi}^!_m: \tilde{K}^{*+1}(P(m-1)) \to \tilde{K}^{*+1}(Q * \ldots * Q)$ factors through $\tilde{K}^{*+1}(Q * \ldots * Q, Q) \to \tilde{K}^{*+1}(Q * \ldots * Q)$. Then by $\hat{\pi}^!_m(u_i) = 0$, it follows that the image of u_i by the homomorphism $\tilde{K}^{*+1}(P(m-1)) \to \tilde{K}^{*+1}(Q * \ldots * Q, Q)$ induced by $\hat{\pi}_m$ can be written as $\delta(x')$, where $x' \in \tilde{K}^*(Q)$ and δ is the connecting homomorphism $K^*(Q) \to$


K̃*(X) = K̃*(X) = K̃*(X)
K̃*(Q) = K̃*(Q)
K̃*+1(Q* . . . *Q, Q) ← K̃*+1(C(Q* . . . *Q), Q) → K̃*+1(CQ, Q)
K̃*+1(X* . . . *X, X) ← K̃*+1(C(X* . . . *X), X) → K̃*+1(CX, X)
K̃*+1(E^k(X), X) ← K̃*+1(CE^k(X), X) → K̃*+1(CX, X)
≅
h_m^!
(Ch_m)!

$K^{*+1}(Q*\ldots*Q, Q)$. Since $j^!$ is surjective, we can choose $x \in P$ such that $j^!(x) = x'$. Then by the commutativity of the diagram opposite, together with that of [**7**, (3.2)], it follows that the restriction to ΣQ of u_i coincides with $s^!(x')$. On the other hand, the restriction to ΣX of u_i is $s^!(x_i)$ by the choice of u_i. Since the restriction of $j^!$ to P is injective, we obtain that $x = x_i$ and $x' = j^!(x_i)$. This implies the proposition.

We prepare one more proposition to determine the image of the connecting homomorphism $\bar{\delta}_m$ for the exact sequence associated with the triad mapping $\hat{\pi}_m$.

PROPOSITION 2.2 *Each element of $P \otimes \ldots \otimes P \subseteq K^*(E^{m-1})$ is primitive with respect to $\hat{\pi}_m|_{Q\times(Q*\ldots*Q)}$ in the sense of* [**16**]. *The respective projections of e to Q and $Q*\ldots*Q$ are given by x and e, respectively, where $x = v^!(e)$ for $m = 2$, 0 for $m \geqq 3$; v is the homotopy inversion of the H-space X and we regard $P\otimes\ldots\otimes P$ as $\tilde{K}^*(Q*\ldots*Q)$ by the homomorphism $(j*\ldots*j)^! \circ h^!_{m-1}: \tilde{K}^*(\hat{E}^{m-1}) \to \tilde{K}^*(Q*\ldots*Q)$.*

Proof. In the case $m = 2$, by Proposition 1.1, (1.2) and (1.3), it follows that $\hat{\pi}_2|_{Q\times Q} = \pi_2 \circ h_2 \circ (j \times j)$, $\pi_2 \circ h_2|_{X\times X} = \hat{\mu}_1$, $\hat{\mu}_1|_{X\times\{*\}} = v$ and $\hat{\mu}_1|_{\{*\}\times X} = id$. By dimensional arguments, it follows that $\mu_1^!(x_i) - v^!(x_i)\otimes 1 - 1 \otimes x_i$ lies in $D \otimes \tilde{K}^*(X) \oplus \tilde{K}^*(X) \otimes D \subseteq \operatorname{Ker}(j\times j)^!$. In the case $m \geqq 3$, similarly we obtain that $\hat{\pi}_m|_{Q\times(Q*\ldots*Q)} = \hat{\mu}_{m-1}\circ(j\times(h_{m-1}\circ(j*\ldots*j)))$, $\hat{\mu}_{m-1}|_{X\times\{*\}} \sim *$ and $\hat{\mu}_{m-1}|_{\{*\}\times E^{m-1}} \sim id$. Again by dimensional arguments, it follows that $\hat{\mu}^!_{m-1}(e) - 1 \times e$ lies in $\hat{S}_m \subseteq \operatorname{Ker}(j\times(j*\ldots*j))^!$. This completes the proof of Proposition 2.2.

By using the exact sequence induced by $\hat{\pi}_m$, we obtain the following short exact sequence of R-modules:

$$0 \to \operatorname{Coker}\hat{\pi}^!_m \to K^*(Q(m)) \to \operatorname{Ker}\hat{\pi}^!_m \to 0,$$

where $\operatorname{Coker}\hat{\pi}^!_m$ is isomorphic to $\operatorname{Im}\bar{\delta}_m$ by the connecting homomorphism $\bar{\delta}_m$, $\operatorname{Ker}\hat{\pi}^!_m = R^{[m]}[u_1, \ldots, u_r] \oplus \hat{S}'_m$ as R-modules and $\hat{S}'_m = S_m \cap \operatorname{Ker}\hat{\pi}^!_m$.

Let us determine $\operatorname{Coker}\hat{\pi}^!_m$ and its image in $K^*(Q(m))$. By (1.3), we may regard $\operatorname{Im}\hat{\pi}^!_m$ as $(P\otimes\ldots\otimes P)\cap\operatorname{Im}\pi^!_m$. Also $(P\otimes\ldots\otimes P)\cap\operatorname{Im}\pi^!_m = (P\otimes\ldots\otimes P)\cap\operatorname{Im}\pi^!_m\circ\delta_{m-1}$, since $(P\otimes\ldots\otimes P)\cup\pi^!_m(u_i) = 0$ by dimensional arguments. By the definition of the Stasheff spectral sequence [**5**, (3.1)], $\pi^!_m\circ\delta_{m-1}$ gives the first differential d_1. Then by the proof of [**7**, Proposition 3.6] in the case $m \geqq 3$, and by dimensional arguments in the case $m = 2$, $(\underbrace{P\otimes\ldots\otimes P}_{m})\cap\operatorname{Im} d_1$ is the module generated by the set $\{x_{i_1}\otimes\ldots\otimes(x_{i_j}\otimes x_{i_{j+1}} - x_{i_{j+1}}\otimes x_{i_j})\otimes\ldots\otimes x_{i_m}\}$. Thus we obtain that

$$\begin{aligned}\operatorname{rank}_R \operatorname{Coker}\hat{\pi}^!_m &= r(r+1)\ldots(r+m-1)/m! \\ &= \#\{(i_1, \ldots, i_m);\ 1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_m \leqq r\}.\end{aligned}$$

On the other hand, by [**16**, Corollary 1.4] and Proposition 2.2, it follows that $\operatorname{Im}\delta_{m-1}$ is generated by the set $\{u_{i_1}\ldots u_{i_m};\ 1\leqq i_1 \leqq \ldots \leqq i_m \leqq r\} \subseteq \tilde{K}^*(Q(m))$ by using the Chern character similarly to [**7**, Proposition 4.2]. Hence we obtain the following short exact sequence of R-algebras as R-modules:

$$0 \to M \to K^*(Q(m)) \to \hat{S}_m \to 0,$$

where $M \cong R^{[m+1]}[u_1, \ldots, u_r]$ is the polynomial algebra truncated at height $m+1$ and is a subalgebra of $K^*(Q(m))$, and $\hat{S}_m$ is the pullback of $\hat{S}'_m \subseteq S_{m-1}$, which is a free R-module since R is a principal ideal domain. Hence we obtain $K^*(Q(m)) \cong M \oplus \hat{S}_m$. Clearly the ring structure of M does not depend on the choice of u_i's. By the multiplicative property of Adams operations ψ^k, the module of decomposables in $K^*(Q(m))$ is closed under the action of Adams operations ψ^k. So we have obtained (1), (2) and (5) of Theorem 0.1. To determine the complete ring structure, we prepare

PROPOSITION 2.3. *Let δ_{m-1}, Δ_m and $\bar{\Delta}_m$ be the Mayer–Vietoris coboundary for $XP(m-1)$, $E^m(X)$ and $Q * \ldots * Q$ and $\delta_{m-1}(w) \in \hat{S}'_m$, where $w \in \tilde{S}_m$. Then* $(\pi_m|_{Q\times(Q*\ldots*Q)})^!(w) = 0$.

Proof. From the commutativity of [7, (2.2)] together with (1.3), we obtain

$$\bar{\Delta}_m \circ (\hat{\pi}_m|_{Q\times(Q*\ldots*Q)})^!(w) = (j * \ldots * j)^! \circ h^!_m \circ \Delta_m \circ (\pi_m|_{X\times E^{m-1}(X)})^!$$
$$= \hat{\pi}_m \circ \delta_{m-1}(w) = 0,$$

since $\delta_{m-1}(w)$ is in $\hat{S}'_m \subseteq \operatorname{Ker} \hat{\pi}^!_m$. By [7, (2.1)], this implies$(\hat{\pi}_m|_{Q\times(Q*\ldots*Q)})^!(w) = x \times 1 + 1 \times e$ for some $x \in \tilde{K}^*(Q)$ and $e \in \tilde{K}^*(Q * \ldots * Q)$. On the other hand, using Proposition 1.1 and Remark 1.2, we obtain

$$\begin{aligned}
&(\hat{\pi}_m \mid Q \times (Q * \ldots * Q))^!(w) \\
&\qquad = (j \times j * \ldots * j))^! \circ (h_m|_{X\times X*\ldots*X)})^! \circ (\pi_m|_{X\times E^{m-1}(X)})^!(w) \\
&\qquad = (j \times (j * \ldots * j))^! \circ (h_m|_{X\times(X*\ldots*X)})^!(x' \times 1 + 1 \times w + \Sigma u_a \times v_a) \\
&\qquad = j^!(x') \times 1 + 1 \times (j * \ldots * j)^! \circ h^!_{m-1}(w) + \Sigma u'_a \times v'_a,
\end{aligned}$$

where $u_a \times v_a \in \tilde{K}^*(X) \otimes K^*(E^{m-1}(X))$ and $u'_a \times v'_a \in \tilde{K}^*(Q) \otimes K^*(Q * \ldots * Q)$. Let us recall that w is in $\tilde{S}_{m-1} = \operatorname{Ker}(j * \ldots * j)^! \circ h^!_{m-1}$ and the inclusion $X \to E^{m-1}(X)$ is null-homotopic for $m \geqq 3$. In the case $m = 2$, by similar considerations as in the proof of Proposition 2.2, we obtain that $x' = v'(w) \in D = \operatorname{Ker} j^!$. So we obtain $x = j^!(x') = 0$ and $e = (j * \ldots * j)^! \circ h^!_{m-1}(w) = 0$. This implies the proposition.

This fact means, in the sense of [16], every $w \in S'_m$ is primitive with respect to $\pi_m|_{Q\times(Q*\ldots*Q)}$ and the respective projections are null. Therefore we obtain

$$\hat{S}_m \cdot u_i = 0 \quad \text{for} \quad 1 \leqq i \leqq r.$$

Hence $\hat{S}_m$ is actually an ideal of $K^*(Q(m))$. On the other hand, an element of $\hat{S}_m$ is a transgression image of 0, since the restriction to ΣQ is clearly 0. Hence by Proposition 2.3, we obtain that $\hat{S}_m \cdot \hat{S}_m = 0$ and therefore obtain that $\hat{S}_m \cdot \tilde{K}^*(Q(m)) = 0$. This implies (3), because $\tilde{K}^*(Q(m)) \cdot \tilde{K}^*(Q(m)) = \bar{M} \cdot \bar{M}$. Also this, together with the fact that $\hat{S}_m \cong \hat{S}'_m \subset S_{m-1}$, implies (4). Hence the isomorphism $K^*(Q(m)) \cong M \oplus \hat{S}_m$ gives the complete description of the ring structure of $K^*(Q(m))$. This implies Theorem 0.1.

3. Proofs of Corollary 0.2 and Corollary 0.3

By Theorem 0.1(1), we can choose a system of generators of M to include the pull-backs of generators which are included in L. Then we put L' to be the subalgebra span by all such pull-backs in M. Then it follows that $L' \subset M$ and the restriction to $P(m-1)$ of L' is L. Then by (5) and (3) of Theorem 0.1, it follows that $\psi^k(L') \subset L' + \mathrm{Ker}\{K^*(Q(m)) \to K^*(P(m-1))\} \subseteq L' + \bar{M} \cdot \bar{M}$. This implies Corollary 0.2.

Next, we show Corollary 0.3. Let $\bar{P}(p-1)$ be the projective $(p-1)$-space of $S^{2n-1}_{(p)}$. Then by [**8**, Theorem 3.1], $\Sigma f: \Sigma X \to S^{2n-1}_{(p)} = S^{2n}_{(p)} \subset \bar{P}(p-1)$ can be extendable to $P(p-1)$ the projective $(p-1)$-space of X, say $\bar{f}: \bar{P}(p-1) \to P(p-1)$. It is known that $K^*(\bar{P}(p-1)) \cong R^{[p]}[z]$, where z is an extension of $s^!(y)$ to $\bar{P}(p-1)$ and $R = \mathbb{Z}_{(p)}$. Then it follows that $u = \bar{f}^!(z)$ is an extension of $\Sigma f!(s!(y))$ and $\psi^k z - k^n z$ is decomposable for all k. Hence $\psi^k u - k^n u$ is decomposable and u is a generator corresponding to x. Then by Corollary 0.2, we can choose a generator u' in $K^*(Q(m))$ such that the restriction to $P(m-1)$ of u' is u and $\psi^k u' - k^n u' \in \bar{M} . \bar{M}$. This implies Corollary 0.3.

4. Proof of Proposition 0.5

By the hypothesis of Proposition 0.5, the $\mathbb{F}_2$-cohomology of X is an exterior algebra on odd dimensional generators. Since $\pi_2^*: H^*(\Sigma X; \mathbb{F}_2) \to H^*(Q * Q; \mathbb{F}_2)$ is injective on the module generated by the elements $s^*(x_i x_j)$ where s^* denotes the suspension isomorphism and x_i's are odd generators, it follows that

$$\hat{T}_2^{2i} = 0 \quad \text{for} \quad i \leqq 3d-2, \tag{4.1}$$

$$\hat{T}_2^{2i+1} = 0 \quad \text{for} \quad i \leqq 4d-2; \qquad \hat{T}_2^k = 0 \quad \text{for} \quad k \leqq 6d-3. \tag{4.2}$$

By Corollary 0.2, we may choose $N \subseteq H^*(Q(2); \mathbb{F}_2)$ such as

$$Sq^{2i}(N) \subset N,$$

where N is a polynomial algebra truncated at height 3. Hence N satisfies the condition of [**18,** Theorem 1.4]. This yields the description of the action of Steenrod squares. Then by using Adams secondary operations on $Q(2)$ (rather than $P(2)$) with (4.1) and (4.2), we obtain Proposition 0.5 by the arguments given in [**17**].

5. Proof of Corollary 0.6

From (Hubbuck, pers. comm.) if a space Y has no torsion in its homology, there is a Kronecker product $\langle\ ,\ \rangle: \tilde{K}_i(Y) \times \tilde{K}^i(Y) \to R$. This enables us to dualise the action of Adams operations ψ^k for all k, where we regard the action on $\tilde{K}^1(Y)$ as the action on $\tilde{K}^0(\Sigma Y)$. Since the induced right action on $K_*(Y)$ of ψ^k is determined uniquely by the duality $\langle\ ,\ \rangle$, the action is natural, that is, commutes with the homomorphism induced from a mapping between such spaces.

Let $G_n = U(n)$ or $Sp(n)$. By [**10**], there is a generating variety $Q_n \subset G_n$ with $Q_n \cap G_k = Q_k$. Hence, there is an inclusion from the collapsing space Q_{n-1}/Q_{k-1} to the Stiefel manifold $O_{n,k} = G_n/G_k$. This inclusion $Q_{n-1}/Q_{k-1} \subset G_{n,k}$ gives a generating subspace of $G_{n,k}$ in our sense by [**10**, Proposition 3.8] using

Atiyah–Hirzebruch spectral sequence. Hence we obtain $Q(2)$ for $O_{n,k}$:

$$\begin{array}{ccc} G_n & \longleftarrow & Q_n \\ \downarrow & & \downarrow \\ O_{n,k} & \overset{j}{\longleftarrow} & Q_{n,k} \end{array} \tag{5.1}$$

The multiplication $\mu: G_n \times G_n \to G_n$ induces an action $\phi: G_n \times O_{n'k} \to O_{n,k'}$ by **[10]**. Then $K_*(G_n)$ is an exterior algebra and $K_*(0_{n,k})$ is a quotient module of $K_*(G_n)$. Let D be the module of decomposables in $K_*(G_n)$. Then it follows that

$$\tilde{K}_*(G_n) \cong D \oplus j_!(\tilde{K}_*(Q_n)).$$

Then by the commutativity of (5.1), it follows that

$$\tilde{K}_*(O_{n,k}) \cong \pi_!(D) \oplus j_!(\tilde{K}_*(Q_{n,k})), \tag{5.2}$$

since $O_{n,k}$ has no torsion in its homology and π induces a surjection in homology. We remark here that the spaces above have no torsion in their homology. Then it follows that

$$\begin{aligned} &(D)\psi^k \subset D, \\ &(\pi_!(D))\psi^k \subset \pi_!(D), \\ &(j_!(\tilde{K}_*(Q_{n,k})))\psi^k \subset j_!(\tilde{K}_*(Q_{n,k})). \end{aligned}$$

We put L to be the dual of $j_!(\tilde{K}_*(Q_{n,k}))$ which annihilates $\pi_!(D)$. Then by (5.2) and the duality, it follows that

$$\psi^k(L) \subset L \quad \text{for all } k.$$

By Corollary 0.2, we can choose M in $K^*(Q(2))$ such that $\psi^k(M) \subset M$. Then by replacing $XP(2)$ in **[3]** with our $Q(2)$, we obtain Corollary 0.6 using the arguments given in **[3]** and Proposition 0.5.

6. Proofs of Proposition 0.7 and Corollary 0.8

Clearly the wedge sum $Q = S^{n_1} \vee \ldots \vee S^{n_r}$ gives a generating subspace of $X = S^{n_1} \times \ldots \times S^{n_r}$. Since E^m has a $(\mathrm{mod}\, p)$ homotopy type of wedge sum of spheres, we can choose a spherical module basis of $\tilde{K}^*(E^m)$ and S_{m-1}. The restrictions u_i' of generators of M to $P(p-1)$ satisfy $\psi^k u_i' - k^{n_i} u_i' \in S_{p-1} \oplus \bar{M} \cdot \bar{M}$, and u_i' has the exact filtration degree n_i. Then by dimensional arguments, it follows that

$$\psi^k(\pi_p^!(u_i')) - k^{n_i}\pi_p^!(u_i') \in \pi_p^!(S_{p-1}) \cap \tilde{S}_p, \tag{6.1}$$

where $\tilde{S}_m = \sum_j \tilde{K}^*(X) \otimes \ldots \otimes D \otimes \ldots \otimes \tilde{K}^*(X)$ and D is the module of decomposables. We choose v_j in $K^*(E^p)$ such as

$$\pi_p^!(u_i') = a_0 v_0 + \sum_{j \geqq 1} a_j v_j, \tag{6.2}$$

where $\psi^k(v_j) = k^{n_i+j} v_j$ for all k. Then by (6.2), it follows that

$$\psi^k(\pi_p^!(u_i')) = k^{n_i}\pi_p^!(u_i') + \sum_{j \geqq 1} k^{n_i}(k^j - 1) a_j v_j,$$

for all k. Then by (6.1), it follows that

$$\sum_{j\geqq 1} k^{n_i}(k^j-1)a_jv_j \in \pi_p^!(S_{p-1}) \cap \tilde{S}_p. \tag{6.3}$$

We also decompose S_{p-1} as the direct sum of ψ^k-eigenvectors such as $S_{p-1} = \sum_a S_{p-1}^{(a)}$, where $\psi^k(w) = k^a w$ for any $w \in S_{p-1}^{(a)}$. Then by (6.3) we obtain that

$$\begin{aligned} k^{n_i}(k^j-1)a_jv_j &\in \pi_p^!(S_{p-1}^{(n_i+j)}) \cap \tilde{S}_p \\ &\subset \pi_p^!(S_{p-1}) \cap \tilde{S}_p. \end{aligned} \tag{6.4}$$

By the proof of [**7**, Proposition 3.6], it follows that $\tilde{S}_p/\pi_p^!(S_{p-1}) \cap \tilde{S}_p$ has no torsion. Hence (6.4) implies that $v_j \in \pi_p^!(S_{p-1}^{(n_i+j)}) \cap \tilde{S}_p$. So we may choose an element $w_j \in S_{p-1}$ such as $\pi_p^!(w_j) = v_j$ and put $u_i = u_i' - \sum_{j\geqq 1} a_j \,.\, w_j$. Then by (6.2), it follows that

$$\pi_p^!(u_i) = a_0v_0$$

and

$$\psi^k(\pi_p^!(u_i)) = k^{n_i}\pi_p^!(u_i) \tag{6.5}$$

for all k.

On the other hand, we obtain that

$$\psi^k(u_i) - k^{n_i}u_i \in S_{p-1} \bmod \bar{M} \cdot \bar{M}.$$

Since $\pi_p^!$ is injective on S_{p-1} (see [**7**, 4]), it follows, by (6.5), that $\psi^k(u_i) = k^{n_i}u_i \bmod \bar{M} \cdot \bar{M}$. Hence by (1) and (2) of Theorem 0.1, we may assume that M is generated by u_i's. Then by Corollary 0.2, it follows that $\psi^k(M) \subseteq M$ for all k. Hence we can apply the arguments of the proof of the theorem of [**5**] to our M and N. Then we obtain part (i) of Proposition 0.7.

We will show part (ii) by refining the arguments of [**5**]. We fix i; $1 \leqq i \leqq \operatorname{rank} N_p$. Then two formulae [**5**, (2.2)] and [**5**, (2.3)], in the case $q = p$ imply the following formula:

$$\sum_{h=1}^{2} k^{(p-h)(p-1)}p^h R_J^h(k) S_J^{p-h}(u_{p,i}) = \lambda_i' p^2 u_{p,i}^p \bmod p^3, \tag{6.6}$$

where $k = p-1$, $\lambda_i \neq 0 \bmod p$ and $\mathbb{Z}\{u_{p,i'}\} = QN_p$. The arguments given after [**5**, (2.3)] show that the only possibility of contributing the elements $u_{p,i'}^p$ by the mapping $R_J^1(k)$, lies on the elements of the form $u_{p,i'}^{p-1}v_1$, where v_1 is in QN_1. By our hypothesis of connectivity, $Q(p)$ is 2-connected and $QN_1 = 0$. Then by (6.6), it follows that

$$p^2R_J^2(k)S_J^{p-2}(u_{p,i}) = p^2u_{p,i}^p \bmod (p^3) + I, \tag{6.7}$$

where I is the submodule of N_{p^2} generated by the independent elements with $u_{p,i'}^p$, $1 \leqq i' \leqq \operatorname{rank} QN_p$. Using the above, we find that the only possibility of contributing the elements $u_{p,i'}^p$ lies on the elements of the form $u_{p,i'}^{p-2} \,.\, v_2$, where v_2 is in QN_2. Then by (6.7), it follows that there exists an element $v_{2,i}$ in QN_2 for each i such that

$$\begin{aligned} S_J^{p-2}(u_{p,i}) &= \lambda_i u_{p,i}^{p-2}v_{2,i} \bmod (p) + I_0, \\ u_{p,i}^{p-2}R_J^2(k)(v_{2,i}) &= u_{p,i}^p \bmod (p) + I, \end{aligned} \tag{6.8}$$

where I_0 is the submodule of $N_{p(p-2)+2}$ generated by the independent elements with $u_{p,i'}^{p-2}v_2$ for $1 \leqq i' \leqq \operatorname{rank} QN_p$ and $v_2 \in QN_2$. Then by (6.8), it follows that

$$R_J^2(k)(v_{2,i}) = u_{p,i}^2 \bmod (p) + I',$$

where I' is the submodule of N_{2p} generated by the independent elements with $u_{p,i'}^2$, $1 \leqq i' \leqq \operatorname{rank} QN_p$. This implies that $\operatorname{rank} H^3(X; \mathbb{Z}) = \operatorname{rank} QN_2 \geqq \operatorname{rank} N_{2p}/I' = \operatorname{rank} QN_p = \operatorname{rank} H^{2p-1}(X; \mathbb{Z})$. This completes the proof of Proposition 0.7.

Let us turn our attention to the case when $p = 3$. The hypothesis of Corollary 0.8 implies that X is homotopy equivalent to $T \times \tilde{X}$ where T is a product of S^1's and $\tilde{X}$ is the universal covering of X. The homotopy associativity of X inherits the universal covering and $\tilde{X}$ satisfies the condition of Proposition 0.7. Hence $\tilde{X}$ has the homotopy type of a product of S^3's and $S^3 \times S^5$'s. Let us recall that $S^1 = U(1)$ and $S^3 = SU(2)$, and that $SU(3)$ is 3-regular and has the mod 3 homotopy type of $S^3 \times S^5$. This implies Corollary 0.8.

7. Proof of Corollary 0.10

Under the hypothesis of Corollary 0.10, we put $Q = S^7 \times \{*\} \subset S^7 \times X$. Then the inclusion $j: Q \subset S^7 \times X$ satisfies the condition that $j^!: QK^*(S^7 \times X) \to \tilde{K}^*(S^7)$ is surjective. Then by Theorem 0.9, it follows that there exists a complex $Q(3, j)$ with properties $K^*(Q(3, j)) \cong M(3, j) \oplus \hat{S}_3(j)$, $M = M(3, j) \cong R^{[4]}[\xi_4, \{\eta_i^t\}]/R^3(j^!)$; $H^*(Q(3, j)) \cong N(3, j) \oplus \hat{T}_3(j)$ and $N = N(3, j) \cong R^{[4]}[x_4, \{y_i^t\}]/R^3(j^*)$, where ξ_4 and η_i^t correspond to generators of $K^*(S^7)$ and $K^*(X)$, respectively: subscripts denote, following [**4**], the exact filtrations.

We may assume that X is simply connected without any loss of generality, since the homotopy associativity inherits the universal cover as well as the Hopf structure. By dimensional arguments, the generators ξ_4 and x_4 are A_4-primitive, that is, these two elements are extendable to $P(3)$. Then we can choose ξ_4 such that $\psi^k(\xi_4) = k^4\xi_4$ in $QM \oplus \hat{S}_3(j)$, since $QM \cong Q\tilde{K}^*(S^7 \times X) \cong \tilde{K}^*(S^7) \oplus QK^*(X)$ and $\operatorname{Im}\{\tilde{K}^*(P(3)) \to \tilde{K}(Q(3, j))\} \cap \hat{S}_3(j)$ is mapped injectively to $\operatorname{Im}\{\tilde{K}^*(P(3)) \to \tilde{K}^*(P(2))\} \cap S_2 = \operatorname{Ker} \pi_3^! \cap S_2 = 0$. We remark here that, for other generators η_i^t, $\psi^k(\eta_i^t)$ are possibly not in M, while $\psi^k(\bar{M} \cdot \bar{M}) \subset \bar{M} \cdot \bar{M}$. In [**4**], the calculations on $\psi^k(\eta_i^t)$ or $\Phi^k(y_i^t)$ are used in those on $\psi^k(\xi_4)$ or $\Phi^k(x_4)$ and hence, are used to calculate $\psi^k(\eta_i^t \cdot \eta_{i'}^{t'}) = \psi^k(\eta_i^t) \cdot \psi^k(\eta_{i'}^{t'})$. Hence we may ignore the part in $\hat{S}_3(j)$ appearing in the description of $\psi^k(\eta_i^t)$. By [**4**, (3.1)], the elements of height 3 except for ξ_4^3 have no contribution to ξ_4, ξ_4^2 and ξ_4^3. Hence the calculations given in [**4**] are all valid for our M or N modulo, the elements of height 3 far from ξ_4^3 or x_4^3. So by the proof of [**4**, Theorem 1.1], we obtain Corollary 0.10.

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