MAYER-VIETORIS SEQUENCE FOR DIFFERENTIABLE/DIFFEOLOGICAL SPACES

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ABSTRACT. The idea of a space with smooth structure is a generalization of an idea of a manifold. K. T. Chen introduced such a space as a differentiable space in his study of a loop space to employ the idea of iterated path integrals [2, 3, 4, 5]. Following the pattern established by Chen, J. M. Souriau [10] introduced his version of a space with smooth structure, which is called a diffeological space. These notions are strong enough to include all the topological spaces. However, if one tries to show de Rham theorem, he must encounter a difficulty to obtain a partition of unity and thus the Mayer-Vietoris exact sequence in general. In this paper, we introduce a new version of differential forms to obtain a partition of unity, the Mayer-Vietoris exact sequence and a version of de Rham theorem in general. In addition, if we restrict ourselves to consider only CW complexes, we obtain de Rham theorem for a genuine de Rham complex, and hence the genuine de Rham cohomology coincides with the ordinary cohomology for a CW complex.

In this paper, we deal with both differentiable and diffeological spaces. A differentiable space is introduced by K. T. Chen [5] and a diffeological space is introduced by J. M. Souriau [10]. Both of them are developed with an idea of a *plot* – a map from a *domain*.

Let $n \geqq 0$. A non-void open set in $\mathbb{R}^n$ is called an *open n-domain* or simply an *open domain* and a compact convex set with non-void interior in $\mathbb{R}^n$ is called a *convex n-domain* or simply a *convex domain*. We reserve the word 'smooth' for 'differentiable infinitely many times' in the ordinary sense. More precisely, a map from an open or convex domain A to an euclidean space is smooth on A, if it is smooth on $\operatorname{Int} A$ in the ordinary sense and all derivatives extend continuously and uniquely to A (see A. Kriegl and P. W. Michor [9]).

Let us explain more about the difficulty to obtain a partition of unity in the theory of differentiable/diffeological spaces. Apparently, if one tries to show it, he must realize that it is not easy to build-up the arguments because of the shortage of differential forms. In fact, we don't know how to manage it in general. So, in this paper, we include more differential forms to make it easier, as is performed in Section 7. But, at the same time, newly included differential forms should not be so many, because we have to show an equivalence in some sense with the original differential forms, if the space is a manifold.

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1. Differentiable/diffeological spaces

Let us recall a concrete site given by Chen [5] (see J. C. Baes and A. E. Hoffnung [1]).

Definition 1.1. *Let* Convex *be the category of convex domains and smooth maps between them. Then* Convex *is a concrete site with Chen's coverage: a covering family on a convex domain is an open covering by interiors of convex domains.*

On the other hand, a concrete site given by Souriau [10] is as follows.

Definition 1.2. *Let* Open *be the category of open domains and smooth maps between them. Then* Open *is a concrete site with the usual coverage: a covering family on an open domain is an open covering by open domains.*

Let Set be the category of sets. A differentiable or diffeological space is as follows.

Definition 1.3 (Differentiable space)**.** *A differentiable space is a pair* $(X, \mathcal{C}_X)$ *of a set* X *and a contravariant functor* $\mathcal{C}_X : \mathsf{Convex} \to \mathsf{Set}$ *such that*

- *(C0) For any* $A \in \mathrm{Obj}(\mathsf{Convex})$*,* $\mathcal{C}_X(A) \subset \mathrm{Hom}_{\mathsf{Set}}(A, X)$*.*
- *(C1) For any* $x \in X$ *and any* $A \in \mathrm{Obj}(\mathsf{Convex})$*,* $\mathcal{C}_X(A) \ni c_x$ *the constant map.*
- *(C2) Let* $A \in \mathrm{Obj}(\mathsf{Convex})$ *with an open covering* $A = \bigcup_{\alpha\in\Lambda} \mathrm{Int}_A B_\alpha, B_\alpha \in \mathrm{Obj}(\mathsf{Convex})$*. If* $P \in \mathrm{Hom}_{\mathsf{Set}}(A, X)$ *satisfies that* $P|_{B_\alpha} \in \mathcal{C}_X(B_\alpha))$ *for all* $\alpha \in \Lambda$*, then* $P \in \mathcal{C}_X(A)$*.*
- *(C3) For any* $A, B \in \mathrm{Obj}(\mathsf{Convex})$ *and any* $f \in \mathrm{Hom}_{\mathsf{Convex}}(B, A)$*,* $\mathcal{C}_X(f) = f^* : \mathcal{C}_X(A) \to \mathcal{C}_X(B)$ *is given by* $f^*(P) = P\circ f \in \mathcal{C}_X(A)$ *for any* $P \in \mathcal{C}_X(A)$*.*

Definition 1.4 (Diffeological space)**.** *A diffeological space is a pair* $(X, \mathcal{D}_X)$ *of a set* X *and a contravariant functor* $\mathcal{D}_X : \mathsf{Open} \to \mathsf{Set}$ *such that*

- *(D0) For any* $U \in \mathrm{Obj}(\mathsf{Open})$*,* $\mathcal{D}_X(U) \subset \mathrm{Map}(U, X)$*.*
- *(D1) For any* $x \in X$ *and any* $U \in \mathrm{Obj}(\mathsf{Open})$*,* $\mathcal{D}_X(U) \ni c_x$ *the constant map.*
- *(D2) Let* $U \in \mathrm{Obj}(\mathsf{Open})$ *with an open covering* $U = \bigcup_{\alpha\in\Lambda} V_\alpha$*,* $V_\alpha \in \mathrm{Obj}(\mathsf{Open})$*. If* $P \in \mathrm{Hom}_{\mathsf{Set}}(U, X)$ *satisfies that* $P|_{V_\alpha} \in \mathcal{D}_X(V_\alpha)$ *for all* $\alpha \in \Lambda$*, then* $P \in \mathcal{D}_X(U)$*.*
- *(D3) For any* $U, V \in \mathrm{Obj}(\mathsf{Open})$ *and any* $f \in \mathrm{Hom}_{\mathsf{Open}}(V, U)$*,* $\mathcal{D}_X(f) = f^* : \mathcal{D}_X(V) \to \mathcal{D}_X(U)$ *is given by* $f^*(P) = P\circ f \in \mathcal{D}_X(V)$ *for any* $P \in \mathcal{D}_X(U)$*.*

From now on, $\mathcal{E}^X : \mathsf{Domain} \to \mathsf{Set}$ stands for either $\mathcal{C}_X : \mathsf{Convex} \to \mathsf{Set}$ or $\mathcal{D}_X : \mathsf{Open} \to \mathsf{Set}$ to discuss about a differentiable space and a diffeological space simultaneously.

Definition 1.5. *A subset* $O \subset X$ *is open if, for any* $P \in \mathcal{E}^X$ $(\mathcal{E} = \mathcal{C}$ *or* $\mathcal{D})$*,* $P^{-1}(O)$ *is open in* $\mathrm{Dom}\, P$*. When any compact subset of* X *is closed, we say* X *is 'weakly-separated'.*

Definition 1.6. *Let $(X,\mathcal{E}^X)$ and $(Y,\mathcal{E}^Y)$ be differentiable/diffeological spaces, $\mathcal{E}=\mathcal{C}$ or $\mathcal{D}$. A map $f:X\to Y$ is differentiable, if there exists a natural transformation of contravariant functors $\mathcal{E}^f:\mathcal{E}^X\to\mathcal{E}^Y$ such that $\mathcal{E}^f(P)=f\circ P$. The set of differentiable maps between X and Y is denoted by $C^\infty_{\mathcal{E}}(X,Y)$ or simply by $C^\infty(X,Y)$. If further, f is invertible with a differentiable inverse map, f is said to be a diffeomorphism.*

Let us summarize the minimum notions from [2, 3, 4, 5, 10, 1, 14, 7, 11, 6, 8] to build up de Rham theory in the category of differentiable or diffeological spaces as follows.

Definition 1.7 (External algebra)**.** *Let $T^*_n=\mathrm{Hom}(\mathbb{R}^n,\mathbb{R})=\bigoplus_{i=1}^{n}\mathbb{R}\,dx_i$, where $\{dx_i\}_{1\leqq i\leqq n}$ is the dual basis to the standard basis $\{e_i\}_{1\leqq i\leqq n}$ of $\mathbb{R}^n$. We denote by $\wedge^*(T^*_n)$ the exterior (graded) algebra on $\{dx_i\}$, where each dx_i is of dimension 1. In particular, we have $\wedge^0(T^*_n)\cong\wedge^*(T^*_0)\cong\mathbb{R}$, $\wedge^p(T^*_n)=0$ if $p<0$ and $\wedge^p(T^*_n)\cong\wedge^{n-p}(T^*_n)$ for any $p\in\mathbb{Z}$.*

The external algebra fits in with our categorical context as the following form.

Definition 1.8. *A contravariant functor $\wedge^p:\mathsf{Domain}\to\mathsf{Set}$ is given as follows:*

(1) $\wedge^p(A)=\mathrm{Hom}_{\mathsf{Domain}}(A,\wedge^p(T^*_n))$, for any convex n-domain A,

(2) For a smooth map $f:B\to A$ in Domain, $\wedge^p(f)=f^*:\wedge^p(A)\to\wedge^p(B)$ is defined, for any $\omega=\sum_{i_1<\cdots<i_p}a_{i_1,\cdots,i_p}(\boldsymbol{x})\;dx_{i_1}\wedge\cdots\wedge dx_{i_p}\in\wedge^p(A)$, as

$$f^*(\omega)=\sum_{j_1<\cdots<j_p}b_{j_1,\cdots,j_p}(\boldsymbol{y})\cdot dy_{j_1}\wedge\cdots\wedge dy_{j_p},\;\;\boldsymbol{y}\in V,$$

$$b_{j_1,\cdots,j_p}(\boldsymbol{y})=\sum_{i_1<\cdots<i_p}a_{i_1,\cdots,i_p}(f(\boldsymbol{y}))\cdot\frac{\partial(x_{i_1},\cdots,x_{i_p})}{\partial(y_{j_1},\cdots,y_{j_p})},$$

where $\dfrac{\partial(x_{i_1},\cdots,x_{i_p})}{\partial(y_{j_1},\cdots,y_{j_p})}$ denotes the Jacobian determinant.

Definition 1.9. *A natural transformation $d:\wedge^p\to\wedge^{p+1}$ is given as follows: for any domain A, $d:\wedge^p(A)\to\wedge^{p+1}(A)$ is defined, for any $\eta=a(\boldsymbol{x})\,dx_{i_1}\wedge\cdots\wedge dx_{i_p}\in\wedge^p(A)$, as*

$$d\eta=\sum_i\frac{\partial\,a_{i_1,\cdots,i_p}}{\partial\,x_i}(\boldsymbol{x})\,dx_i\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

Then the naturality is obtained using a strait-forward computation.

A differential form is given in this context as follows.

Definition 1.10. *Let $(X,\mathcal{E}^X)$ be a differentiable or diffeological space, $\mathcal{E}=\mathcal{C}$ or $\mathcal{D}$.*

(general): *A differential p-form on X is a natural transformation $\omega:\mathcal{E}^X\to\wedge^p$ given by $\{\omega_A:\mathcal{E}^X(A)\to\wedge^p(A)\,;\,A\in\mathrm{Obj}(\mathsf{Domain})\}$ of contravariant functors $\mathcal{E}^X$, $\wedge^p:\mathsf{Domain}\to\mathsf{Set}$, in other words, ω satisfies $f^*(\omega_B(P))=f^*\circ\omega_B(P)=\omega_A\circ f^*(P)=\omega_A(P\circ f)$ for any map $f:A\to B$ in Domain and a plot $P\in\mathcal{E}^X(B)$. The set* *of differential p-forms on X is denoted by $\mathcal{A}^p_{\mathcal{E}}(X)$ or simply by $\mathcal{A}^p(X)$. We also denote $\mathcal{A}^*_{\mathcal{E}}(X) = \bigoplus_p \mathcal{A}^p_{\mathcal{E}}(X)$ or by $\mathcal{A}^*(X) = \bigoplus_p \mathcal{A}^p(X)$.*

(with compact support): *A differential p-form with compact support on X is a natural transformation $\omega =: \mathcal{E}^X \to \wedge^p(-)$ with a compact subset $K_\omega \subset X$ such that, for any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X$, we have $\mathrm{Supp}\,\omega_A(P) \subset P^{-1}(K_\omega)$. The set of differential p-forms with compact support on X is denoted by $\mathcal{A}^p_{\mathcal{E}_c}(X)$ or simply by $\mathcal{A}^p_c(X)$. We also denote $\mathcal{A}^*_{\mathcal{E}_c}(X) = \bigoplus_p \mathcal{A}^p_{\mathcal{E}_c}(X)$ or $\mathcal{A}^*_c(X) = \bigoplus_p \mathcal{A}^p_c(X)$.*

Example 1.11. *We have $\mathcal{A}^*(\{*\}) \cong \mathbb{R}$ and $\mathcal{A}^*_c(\{*\}) \cong \mathbb{R}$.*

Definition 1.12 (External derivative)**.** *The external derivative of a differential p-form ω on a differentiable/diffeological space X is a differential p+1-form $d\omega$ given by $(d\omega)_A = d\circ\omega_A$ for any $A \in \mathrm{Obj}(\mathsf{Domain})$. If, further we assume $\omega \in \mathcal{A}^p_c(X)$, we clearly have $d\omega \in \mathcal{A}^{p+1}_c(X)$. Thus the external derivative induces endomorphisms of $\mathcal{A}^*(X)$ and $\mathcal{A}^*_c(X)$.*

The categories of differentiable spaces and of diffeological spaces are denoted respectvely by $\mathsf{Differentiable}$ and $\mathsf{Diffeology}$, which are different from each other (see [11]). By [10], [5] and [1], we know both of them are cartesian closed, complete and cocomplete.

Definition 1.13. *Let $f : (X, \mathcal{E}^X) \to (Y, \mathcal{E}^Y)$ be a differentiable map, $\mathcal{E} = \mathcal{C}$ or $\mathcal{D}$.*

(1) We obtain a homomorphism $f^\sharp : \mathcal{A}^p(Y) \to \mathcal{A}^p(X)$: let $\omega \in \mathcal{A}^p(Y)$. Then $(f^\sharp\omega)_A(P) = \omega_A(f\circ P)$ for any $P \in \mathcal{E}^X(A)$ and $A \in \mathrm{Obj}(\mathsf{Domain})$.

(2) If a differentiable map f is proper, then we have $f^\sharp(\mathcal{A}^p_c(Y)) \subset \mathcal{A}^p_c(X)$ by taking $K_{f^\sharp\omega} = f^{-1}(K_\omega)$ for any $\omega \in \mathcal{A}^p_c(Y)$.

Definition 1.14. *For an inclusion $j : U \hookrightarrow X$ of an open set U into a weakly-separated differentiable/diffeological space X, a homomorphism $j_\sharp : \mathcal{A}^p_c(U) \to \mathcal{A}^p_c(X)$ is defined as follows: for any $\omega \in \mathcal{A}^p_c(U)$, $j_\sharp\omega \in \mathcal{A}^p_c(X)$ is given, for n-domain B and $Q \in \mathcal{E}^X(B)$, by*

$$\begin{cases} (j_\sharp\omega)_B(Q)|_A = \omega_A(Q|_A), & \text{if } A \text{ is an open } n\text{-domain in } Q^{-1}(U), \\ (j_\sharp\omega)_B(Q)|_A = 0, & \text{if } A \text{ is an open } n\text{-domain in } B \smallsetminus Q^{-1}(K_\omega) \end{cases}$$

with $K_{j_\sharp\omega} = K_\omega \subset U \subset X$. Here, $\{Q^{-1}(U), B \smallsetminus Q^{-1}(K_\omega)\}$ is an open covering of B.

Remark 1.15. *In Definition 1.14, the map $j_\sharp$ induced from an inclusion $j : U \hookrightarrow X$ satisfies that $(j_\sharp\omega)_B(j\circ Q) = \omega_B(Q)$ for any $B \in \mathrm{Obj}(\mathsf{Domain})$ and $Q \in \mathcal{E}^U(B)$.*

Proposition 1.16. *There is an isomorphism $\Phi : \mathcal{A}^0(X) \cong C^\infty(X, \mathbb{R})$ such that $\Phi(\omega)\circ f = \Phi(f^\sharp(\omega))$ for any $\omega \in \mathcal{A}^0(X)$ and $f \in C^\infty(Y, X)$.*

Proof: Firstly, we define a homomorphism $\Phi : \mathcal{A}^0(X) \to \mathrm{Hom}_{\mathsf{Set}}(X, \mathbb{R})$ by $\Phi(\omega)(x) = \omega_{\{*\}}(c_x)(*) \in \mathbb{R}$ for any $\omega \in \mathcal{A}^0(X)$ and $x \in X$. By definition, Φ clearly is a homomorphism.

Secondly, we show $\mathrm{Im}\,\Phi \subset C^\infty(X, \mathbb{R})$. For any n-domain A and $P \in \mathcal{E}^X(A)$, we have $\omega_A(P) : A \to \wedge^0(T_n^*) = \mathbb{R}$. Hence for any $\boldsymbol{x} \in A$, we have $P \circ c_{\boldsymbol{x}} = c_x \in \mathcal{E}^X(\{*\})$ where $x = P(\boldsymbol{x}) \in X$, and hence we have $\omega_A(P)(\boldsymbol{x}) = \omega_A(P) \circ c_{\boldsymbol{x}}(*) = \omega_{\{*\}}(P \circ c_{\boldsymbol{x}})(*) = \omega_{\{*\}}(c_x)(*) = \Phi(\omega)(x) = \Phi(\omega) \circ P(\boldsymbol{x})$, $\boldsymbol{x} \in A$. Thus we have $\omega_A(P) = \Phi(\omega) \circ P$ for any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, and hence $\Phi(\omega) : X \to \mathbb{R}$ is a differentiable map. Moreover, for any differentiable map $f : Y \to X$, we have $\Phi(f^\sharp \omega)(x) = (f^\sharp \omega)_{\{*\}}(c_x)(*) = \omega_{\{*\}}(f \circ c_x)(*) = \omega_{\{*\}}(c_{f(x)})(*) = \Phi(\omega) \circ f(x)$, and hence we obtain $\Phi(f^\sharp \omega) = \Phi(\omega) \circ f$.

Thirdly, by the formula $\omega_A(P) = \Phi(\omega) \circ P$ for any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, ω is completely determined by $\Phi(\omega)$, and hence Φ is a monomorphism.

Finally, for any differentiable map $f : X \to \mathbb{R}$, we have a 0-form ω by $\omega_A(P) = f \circ P$ for any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, which also implies $\Phi(\omega) = f$. Thus Φ is an epimorphism, and it completes the proof of the proposition. □

Definition 1.17. *Let $X = (X, \mathcal{E})$ be a differentiable/diffeological space, $\mathcal{E} = \mathcal{C}$ or $\mathcal{D}$.*

de Rham cohomology: $H^p_{\mathcal{E}}(X) = \dfrac{Z^p_{\mathcal{E}}(X)}{B^p_{\mathcal{E}}(X)}$,

where $Z^p_{\mathcal{E}}(X) = \mathrm{Ker}\, d \cap \mathcal{A}^p_{\mathcal{E}}(X)$ and $B^p_{\mathcal{E}}(X) = d(\mathcal{A}^p_{\mathcal{E}}(X))$.

de Rham cohomology with compact support: $H^p_{\mathcal{E}_c}(X) = \dfrac{Z^p_{\mathcal{E}_c}(X)}{B^p_{\mathcal{E}_c}(X)}$,

where $Z^p_{\mathcal{E}_c}(X) = \mathrm{Ker}\, d \cap \mathcal{A}^p_{\mathcal{E}_c}(X)$ and $B^p_{\mathcal{E}_c}(X) = d(\mathcal{A}^p_{\mathcal{E}_c}(X))$.

From now on, we often abbreviate as $H^p(X) = H^p_{\mathcal{E}}(X)$, $H^p_c(X) = H^p_{\mathcal{E}_c}(X)$ and so on.

Remark 1.18. *We have $H^p_{\mathcal{E}_c}(M) \cong H^p_{dR}(M)$ and $H^p_{\mathcal{E}_c}(M) \cong H^p_{dR_c}(M)$ for a manifold M, where we denote by $H^p_{dR}(M)$ ($H^p_{dR_c}(M)$) the de Rham cohomology (with compact support).*

Proposition 1.19. *Let $(X, \mathcal{E}^X)$ and $(Y, \mathcal{E}^Y)$ be differentiable/diffeological spaces.*

(1) For a differentiable map $f : X \to Y$, the homomorphism $f^\sharp : \mathcal{A}^*(Y) \to \mathcal{A}^*(X)$ induces a homomorphism $f^* : H^*(Y) \to H^*(X)$.

(2) If a differentiable map $f : X \to Y$ is proper, then the homomorphism $f^\sharp : \mathcal{A}^*_c(Y) \to \mathcal{A}^*_c(X)$ induces a homomorphism $f^* : H^*_c(Y) \to H^*_c(X)$.

Theorem 1.20. *The de Rham cohomologies determines contravariant functors $H^*_{\mathcal{C}}$: $\mathsf{Differentiable} \to \mathsf{GradedAlgebra}$ and $H^*_{\mathcal{D}} : \mathsf{Diffeology} \to \mathsf{GradedAlgebra}$.*

Proposition 1.21. *Let $(X, \mathcal{E}^X)$ be a weakly-separated differentiable/diffeological space and U an open set in X. Then the homomorphism $j_\sharp : \mathcal{A}^*_c(U) \to \mathcal{A}^*_c(X)$ induced from the canonical inclusion $j : U \hookrightarrow X$ induces a homomorphism $j_* : H^*_c(U) \to H^*_c(X)$.*

Theorem 1.22 ([5], [10]). *If two differentiable maps $f_0,\ f_1 : X \to Y$ between differentiable/diffeological spaces are homotopic in $C^\infty_{\mathcal{E}}(X,Y)$, $\mathcal{E} = \mathcal{C}$ or $\mathcal{D}$, i.e., there is a differentiable map $f : I \to C^\infty_{\mathcal{E}}(X,Y)$ such that $f(t) = f_t$, $t = 0,1$, then we obtain*

$$f_0^* = f_1^* : H^*_{\mathcal{E}}(Y) \to H^*_{\mathcal{E}}(X).$$

Theorem 1.23. *By definition, we clearly have $H^*_{\mathcal{E}}(\coprod_\alpha X_\alpha) = \prod_\alpha H^*_{\mathcal{E}}(X_\alpha)$, $\mathcal{E} = \mathcal{C}$ or $\mathcal{D}$.*

Example 1.24. *For a differentiable/diffeological space $(\{*\},\mathcal{E}^*)$ with $\mathcal{E}^*(A) = \{c_*\}$ for any $A \in \mathrm{Obj}(\mathsf{Domain})$, we have $H^0(X) = \mathcal{A}^0(X) = \mathbb{R}$ and $H^p(X) = \mathcal{A}^p(X) = 0$ if $p \neq 0$.*

2. Mayer-Vietoris sequence for differentiable spaces

Definition 2.1 (partition of unity). *Let $(X,\mathcal{E}^X)$ be a differentiable/diffeological space and $\mathcal{U}$ an open covering of X. A set of 0-forms $\boldsymbol{\rho} = \{\rho^U\,;\,U \in \mathcal{U}\}$ is called a partition of unity belonging to $\mathcal{U}$, if, for any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, $\mathrm{Supp}\,\rho^U_A(P) \subset P^{-1}(U)$ and $\sum_{U\in\mathcal{U}} \rho^U_A(\boldsymbol{x}) = 1$, $\boldsymbol{x} \in A$. If further there is a family $\{G_U\,;\,U \in \mathcal{U}\}$ of closed sets in X such that, $\mathrm{Supp}\,\rho^U_A(P) \subset P^{-1}(G_U)$ for any A and P above, then we say that $\boldsymbol{\rho}$ is 'normal'.*

The above definition of a partition of unity using the notion of 0-form first appeared in Izumida [8] which was essentially the same as the one in Haraguchi [6] using the notion of a differentiable function, since a differential 0-form is a differentiable function, if we adopt the usual definition of 0-form. We introduce a special kind of open coverings as follows.

Definition 2.2 (Nice covering). *Let X be a differentiable space. An open covering $\mathcal{U}$ of X is nice, if there is a partition of unity $\{\rho^U_A : A \to I = [0,1]\,;\,U \in \mathcal{U}\}$ belonging to $\mathcal{U}$, i.e., $\{\rho^U\}$ are differential 0-forms with $\mathrm{Supp}\,\rho^U_A(P) = \mathrm{Cl}\,(\rho^U_A(P)^{-1}(I \smallsetminus \{0\})) \subset P^{-1}(U)$, $U \in \mathcal{U}$ satisfying $\sum_{U\in\mathcal{U}} \rho^U_A(P)(\boldsymbol{x}) = 1$ for any $\boldsymbol{x} \in A$, where $\rho^U_A(P)(\boldsymbol{x}) \neq 0$ for finitely many U.*

Theorem 2.3 (see [6] or [8]). *Let $\mathcal{U} = \{U_1, U_2\}$ be a nice open covering of a differentiable/diffeological space $(X,\mathcal{E}^X)$ with a partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to $\mathcal{U}$. Then $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1,2$, induce homomorphisms $\psi^\natural : \mathcal{A}^p(X) \to \mathcal{A}^p(U_1) \oplus \mathcal{A}^p(U_2)$ and $\phi^\natural : \mathcal{A}^p(U_1) \oplus \mathcal{A}^p(U_2) \to \mathcal{A}^p(U_1 \cap U_2)$ by $\psi^\natural(\omega) = i_1^\sharp\omega \oplus i_2^\sharp\omega$ and $\phi^\natural(\eta_1 \oplus \eta_2) = j_1^\sharp\eta_1 - j_2^\sharp\eta_2$, and the following sequence is exact.*

$$\begin{aligned} H^0(X) \to \cdots \to H^p(X) &\xrightarrow{\psi^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{\phi^*} H^p(U_1 \cap U_2) \\ &\to H^{p+1}(X) \xrightarrow{\psi^*} H^{p+1}(U_1) \oplus H^{p+1}(U_2) \xrightarrow{\phi^*} H^{p+1}(U_1 \cap U_2) \to \cdots, \end{aligned}$$

where ψ^* and ϕ^* are induced from $\psi^\natural$ and $\phi^\natural$.

Proof: Let $U_0 = U_1 \cap U_2$. We show that the following sequence is short exact.

$$0 \longrightarrow \mathcal{A}^p(X) \xrightarrow{\psi^\natural} \mathcal{A}^p(U_1) \oplus \mathcal{A}^p(U_2) \xrightarrow{\phi^\natural} \mathcal{A}^p(U_0) \longrightarrow 0.$$

(exactness at $\mathcal{A}^p(X)$): Assume $\psi^\natural(\omega) = 0$, and so $j_t^\sharp\omega = 0$ for $t = 1, 2$. For any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, we define $P_t : P^{-1}(U_t) \to U_t$, $t{=}1,2$ by $P_t(\boldsymbol{x}) = P(\boldsymbol{x})$ for any $\boldsymbol{x} \in P^{-1}(U_t)$, so that $P|_{P^{-1}(U_t)} = j_t{\circ}P_t$ for $t{=}1,2$. Then, for any $\boldsymbol{x} \in A$, there is an open subset $A_{\boldsymbol{x}} \in \mathrm{Obj}(\mathsf{Domain})$ of A such that $\boldsymbol{x} \in A_{\boldsymbol{x}} \subset P^{-1}(U_t)$ for $t{=}1$ or 2. In each case, we have $\omega_A(P)|_{A_{\boldsymbol{x}}} = \omega_{A_{\boldsymbol{x}}}(P|_{A_{\boldsymbol{x}}}) = \omega_{A_{\boldsymbol{x}}}(P|_{P^{-1}(U_t)}|_{A_{\boldsymbol{x}}}) = \omega_{A_{\boldsymbol{x}}}(j_t{\circ}P_t|_{A_{\boldsymbol{x}}}) = (j_t^\sharp\omega)_{A_{\boldsymbol{x}}}(P_t|_{A_{\boldsymbol{x}}}) = 0$, and hence $\omega_A(P)|_{A_{\boldsymbol{x}}} = 0$ for any $\boldsymbol{x} \in A$. Thus $\omega_A(P) = 0$ for any A and P, which implies that $\omega = 0$. Thus $\psi^\natural$ is monic.

(exactness at $\mathcal{A}^p(U_1) \oplus \mathcal{A}^p(U_2)$): Assume $\phi^\natural(\eta^{(1)}{\oplus}\eta^{(2)}) = 0$, and so $i_1^\sharp\eta^{(1)} = i_2^\sharp\eta^{(2)}$. Then we construct $\omega \in \mathcal{A}^p(X)$ as follows. For any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, $\{P^{-1}(U_t)\,;\, t{=}1,2\}$ is an open covering of A, and for $t{=}0,1,2$ we obtain $P_t : P^{-1}(U_t) \to U_t$ given by $P_t(\boldsymbol{x}) = P(\boldsymbol{x})$ for any $\boldsymbol{x}{\in}P^{-1}(U_t)$, so that $P_t|_{P^{-1}(U_0)} = i_t{\circ}P_0$ for $t{=}1,2$. For any $\boldsymbol{x} \in A$, there is an open subset $A_{\boldsymbol{x}} \in \mathrm{Obj}(\mathsf{Domain})$ of A such that $\boldsymbol{x} \in A_{\boldsymbol{x}} \subset P^{-1}(U_t)$ for $t = 1$ or 2. Using it, we define $\omega_A(P)(\boldsymbol{x}) = \eta^{(t)}_{A_{\boldsymbol{x}}}(P|_{A_{\boldsymbol{x}}})(\boldsymbol{x})$ for any $\boldsymbol{x} \in A$. In case when $A_{\boldsymbol{x}}{\subset}A_0{=}A_1{\cap}A_2$, we have $\eta^{(1)}_{A_{\boldsymbol{x}}}(P_1|_{A_{\boldsymbol{x}}}) = \eta^{(1)}_{A_{\boldsymbol{x}}}(i_1{\circ}P_0|_{A_{\boldsymbol{x}}}) = (i_1^\sharp\eta^{(1)})_{A_{\boldsymbol{x}}}(P_0|_{A_{\boldsymbol{x}}}) = (i_2^\sharp\eta^{(2)})_{A_{\boldsymbol{x}}}(P_0|_{A_{\boldsymbol{x}}}) = \eta^{(2)}_{A_{\boldsymbol{x}}}(i_2{\circ}P_0|_{A_{\boldsymbol{x}}}) = \eta^{(2)}_{A_{\boldsymbol{x}}}(P_2|_{A_{\boldsymbol{x}}})$, and hence $\eta^{(1)}_{A_{\boldsymbol{x}}}(P_1|_{A_{\boldsymbol{x}}}) = \eta^{(2)}_{A_{\boldsymbol{x}}}(P_2|_{A_{\boldsymbol{x}}})$. It implies that ω is well-defined and $\psi^\natural(\omega) = \eta^{(1)}{\oplus}\eta^{(2)}$. The converse is clear and we obtain $\mathrm{Ker}\,\phi^\natural = \mathrm{Im}\,\psi^\natural$.

(exactness at $\mathcal{A}^p(U_0)$): Assume $\kappa \in \mathcal{A}^p(U_0)$. Then we define $\kappa^{(t)} \in \mathcal{A}^p(U_t)$, $t{=}1,2$ defined as follows. For any $A_t \in \mathrm{Obj}(\mathsf{Domain})$ and a plot $P_t : A_t \to U_t$, we define $\kappa^{(t)}_{A_t}(P_t)(\boldsymbol{x})$ by $(-1)^{t-1}\rho^{3-t}_{A_t}(P_t)(\boldsymbol{x}){\cdot}\kappa_{A_t}(P_t)(\boldsymbol{x})$ if $\boldsymbol{x} \in P_t^{-1}(U_{3-t})$ and by 0 if $\boldsymbol{x} \notin P_t^{-1}(\mathrm{Supp}\,\rho^{3-t}_{A_t}(P_t))$. Then we see that $\kappa^{(t)}$ is well-defined differential p-form on U_t and $i_1^\sharp\kappa^{(1)}{-}i_2^\sharp\kappa^{(2)} = \kappa$, and hence $\phi^\natural(\kappa^{(1)}{\oplus}\kappa^{(2)}) = \kappa$. Thus $\phi^\natural$ is an epimorphism.

Since $\psi^\natural$ and $\phi^\natural$ are clearly cochain maps, we obtain the desired long exact sequence. □

Let us turn our attention to the differential forms with compact support.

Theorem 2.4 (see [6] or [8]). *Let $(X, \mathcal{E}^X)$ be a weakly-separated differentiable/diffeological space and $\mathcal{U} = \{U_1, U_2\}$ a nice open covering of X with a normal partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to $\mathcal{U}$. Then $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce homomorphisms $\phi_\natural : \mathcal{A}^p_c(U_1 \cap U_2) \to \mathcal{A}^p_c(U_1){\oplus}\mathcal{A}^p_c(U_2)$ and $\psi_\natural : \mathcal{A}^p_c(U_1){\oplus}\mathcal{A}^p_c(U_2) \to \mathcal{A}^p_c(X)$ by $\phi_\natural(\omega) = i_{1\sharp}\omega{\oplus}i_{2\sharp}\omega$ and $\psi_\natural(\eta_1{\oplus}\eta_2) = j_{1\sharp}\eta_1{-}j_{2\sharp}\eta_2$, and the following sequence is exact.*

$$H^0_c(U_1 \cap U_2) \to \cdots \to H^p_c(U_1 \cap U_2), \xrightarrow{\phi_*} H^p_c(U_1){\oplus}H^p_c(U_2) \xrightarrow{\psi_*} H^p_c(X)$$
$$\to H^{p+1}_c(U_1 \cap U_2) \xrightarrow{\phi_*} H^{p+1}_c(U_1){\oplus}H^{p+1}_c(U_2) \xrightarrow{\psi_*} H^{p+1}_c(X) \to \cdots,$$

where ψ_* and ϕ_* are induced from $\psi_\natural$ and $\phi_\natural$.

Proof: Let $U_0 = U_1 \cap U_2$. We show that the following sequence is short exact.

$$0 \longrightarrow \mathcal{A}^p_c(U_0) \xrightarrow{\phi_\natural} \mathcal{A}^p_c(U_1) \oplus \mathcal{A}^p_c(U_2) \xrightarrow{\psi_\natural} \mathcal{A}^p_c(X) \longrightarrow 0.$$

(exactness at $\mathcal{A}^p_c(U_0)$): Assume $\phi_\natural(\omega) = 0$. Then $i_{1\sharp}(\omega) = i_{2\sharp}(\omega) = 0$. Since $i_{1\sharp}(\omega)$ is an extension of ω, we obtain $\omega = 0$. Thus $\phi_\natural$ is a monomorphism.

(exactness at $\mathcal{A}^p_c(U_1) \oplus \mathcal{A}^p_c(U_2)$): Assume $\psi_\natural(\eta^{(1)} \oplus \eta^{(2)}) = 0$. By definition, we have $j_{1\sharp}(\eta^{(1)}) = j_{2\sharp}(\eta^{(2)})$. For any $A \in \mathrm{Obj}(\mathsf{Domain})$ and $P \in \mathcal{E}^X(A)$, we have $j_{1\sharp}(\eta^{(1)})_A(P) = j_{2\sharp}(\eta^{(2)})_A(P)$. So, for any $B \in \mathrm{Obj}(\mathsf{Domain})$ and a plot $Q : B \to U_0$, $\eta^{(1)}_B(i_1 \circ Q) = j_1^\sharp \eta^{(1)}_B(j_1 \circ i_1 \circ Q) = j_2^\sharp \eta^{(2)}_B(j_2 \circ i_2 \circ Q) = \eta^{(2)}_B(i_2 \circ Q)$. So we define $\eta^{(0)} \in \mathcal{A}^p(U_0)$ by $\eta^{(0)}_B(Q) = \eta^{(1)}_B(i_1 \circ Q) = \eta^{(2)}_B(i_2 \circ Q)$. On the other hand, $K_{j_{t\sharp}\eta^{(t)}} = K_{\eta^{(t)}}$ by definition, and hence we obtain $\mathrm{Supp}\,\eta^{(0)}_B(Q) = \mathrm{Supp}\,\eta^{(1)}_B(i_1 \circ Q) = \mathrm{Supp}\,\eta^{(2)}_B(i_2 \circ Q) \subset Q^{-1}(K_{\eta^{(1)}} \cap K_{\eta^{(2)}})$. Then $\eta^{(0)} \in \mathcal{A}^p_c(U_0)$ for $K_{\eta^{(0)}} = K_{\eta^{(1)}} \cap K_{\eta^{(2)}}$ is compact.

(exactness at $\mathcal{A}^p_c(X)$): Assume $\kappa \in \mathcal{A}^p_c(X)$. For any $A_t \in \mathrm{Obj}(\mathsf{Domain})$ and a plot $P_t : A_t \to U_t$, we define $\kappa^{(t)}_{A_t}(P_t)(\boldsymbol{x})$ by $(-1)^{t-1} \rho^{(t)}_{A_t}(P_t)(\boldsymbol{x}) \cdot \kappa_{A_t}(j_t \circ P_t)(\boldsymbol{x})$ if $\boldsymbol{x} \in P_t^{-1}(U_0)$ and by 0 if $\boldsymbol{x} \notin \mathrm{Supp}\,\rho^{(t)}_{A_t}(P_t)$. Then $\kappa^{(t)}$ is a well-defined differential p-form on U_t with compact support $K_{\kappa^{(t)}} = K_\kappa \cap G_{U_t}$ in U_t and $j_1^\sharp \kappa^{(1)} - j_2^\sharp \kappa^{(2)} = \kappa$, and hence we have $\psi_\natural(\kappa^{(1)} \oplus \kappa^{(2)}) = \kappa$. Thus $\psi_\natural$ is an epimorphism.

Since $\phi_\natural$ and $\psi_\natural$ are clearly cochain maps, we obtain the desired long exact sequence. □

3. Cube Category

Definition 3.1. *A concrete monoidal site $\underline{\square}$ is defined as follows:*

Object: $\mathrm{Obj}(\underline{\square}) = \{\underline{0}, \underline{1}, \underline{2}, \cdots\} \approx \mathbb{N}_0, \quad \underline{n} = \square^n_L := \square^n \cap L,$
where $\square^n = \{(t_1, \ldots, t_n)\,;\, 0 \le t_1, \cdots, t_n \le 1\}$ and $L = \mathbb{Z}^n \subset \mathbb{R}^n$ is an integral lattice.

Morphism: $\mathrm{Hom}_{\underline{\square}}$ *is generated by the following sets of morphisms.*

boundary: $\partial^\epsilon_i : \underline{n} \to \underline{n{+}1}$, $\epsilon \in \dot{I} = \{0, 1\}, 1 \le i \le n{+}1$, $n \in \mathbb{N}_0$, *given by*

$$\partial^\epsilon_i(\boldsymbol{t}) = (t_1, \ldots, t_{i-1}, \epsilon, t_{i+1}, \ldots, t_n) \quad \textit{for} \quad \boldsymbol{t} = (t_1, \cdots, t_n) \in \square^n,$$

degeneracy: $\varepsilon_i : \underline{n{+}1} \to \underline{n}$, $0 \le i \le n$, $n \in \mathbb{N}_0$ *given by*

$$\varepsilon_i(\boldsymbol{t}) = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{n+1}), \quad \boldsymbol{t} = (t_1, \cdots, t_{n+1}) \in \square^n,$$

which satisfies the following relations.

$$(1)\ \ \partial^{\epsilon'}_j \circ \partial^\epsilon_i = \begin{cases} \partial^\epsilon_i \circ \partial^{\epsilon'}_{j-1} & \text{if } i < j \\ \partial^\epsilon_{i+1} \circ \partial^{\epsilon'}_j & \text{if } i \ge j \end{cases} \qquad (2)\ \ \varepsilon_j \circ \varepsilon_i = \begin{cases} \varepsilon_i \circ \varepsilon_{j+1} & \text{if } i \le j \\ \varepsilon_{i-1} \circ \varepsilon_j & \text{if } i > j \end{cases}$$

(3) $\partial^{\epsilon'}_j\circ\varepsilon_i = \begin{cases}\varepsilon_{i+1}\circ\partial^{\epsilon'}_j & \textit{if } i \geq j\\ \varepsilon_i\circ\partial^{\epsilon'}_{j+1} & \textit{if } i < j\end{cases}$ *(4)* $\varepsilon_j\circ\partial^{\epsilon}_i = \begin{cases}\partial^{\epsilon}_{i-1}\circ\varepsilon_j & \textit{if } i > j\\ \partial^{\epsilon}_i\circ\varepsilon_{j-1} & \textit{if } i < j\\ \mathrm{id} & \textit{if } i = j\end{cases}$

Clearly, we can extend and ε_i as smooth maps $\partial^{\epsilon}_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ and $\varepsilon_i : \mathbb{R}^{n+1} \to \mathbb{R}^n$. Let $\underline{\square} : \underline{\square} \to \mathsf{Convex}$ be the covariant functor defined by $\underline{\square}(\underline{n}) = \square^n$, $\underline{\square}(\partial^{\epsilon}_i) = \partial^{\epsilon}_i|_{\square^n} : \square^n \to \square^{n+1}$ and $\underline{\square}(\varepsilon_i) = \varepsilon_i|_{\square^{n+1}} : \square^{n+1} \to \square^n$ and $\underline{\square}(\varepsilon_i) = \varepsilon_i|_{\square^{n+1}} : \square^{n+1} \to \square^n$.

Remark 3.2. *There is a smooth relative homeomorphism* $\pi_n : (\square^n, \partial\square^n) \to (\triangle^n, \partial\triangle^n)$ *given by* $\pi_n(t_1, \cdots, t_n) = (0, s_1, \ldots, s_n, 1),\ s_k = t_k\cdot\cdots\cdot t_n$, *where the standard n-simplex $\triangle^n$ is regarded as* $\triangle^n = \{(s_0, \cdots, s_{n+1}) \in \mathbb{R}^n\,;\, 0 = s_0 \leq s_1 \leq \cdots \leq s_n \leq s_{n+1} = 1\}$.

According to [1], there is a natural embedding ch : $\mathsf{Diffeology} \to \mathsf{Differentiable}$. So, from now on, we deal mainly with differentiable spaces, rather than diffeological spaces. We denote $\mathcal{E}^X_{\underline{\square}} = \mathcal{E}^X\circ\underline{\square}$ and $\wedge^p_{\underline{\square}} = \wedge^p\circ\underline{\square}$, and a plot in $\mathcal{E}^X_{\underline{\square}}(\underline{n}) = \mathcal{E}^X(\square^n)$ is called an n-plot.

Let $X = (X, \mathcal{E}^X)$ be a differentiable space. Then we denote $\Sigma_n(X) = \mathcal{E}^X(\square^n)$ the set of n-plots. Let $\Gamma_n(X)$ be the free abelian group generated by $\Sigma_n(X)$ and $\Gamma^n(X, R) = \mathrm{Hom}(\Gamma_n(X); R)$, where R is a commutative ring with unit. Then $\Gamma^*(X; R)$ is a cochain complex and we obtain a smooth version of cubical singular cohomology $H^*(X, R)$ in a canonical manner, which satisfies axioms of cohomology theories such as additivity, dimension and homotopy axioms together with a Mayer-Vietoris exact sequence.

4. CUBICAL DE RHAM COHOMOLOGY

We introduce a version of a differential form by using $\mathcal{E}^X_{\underline{\square}}$ and $\wedge^p_{\underline{\square}}$.

Definition 4.1 (cubical differential form)**.** *A cubical differential form on a differentiable space X is a natural transformation* $\omega : \mathcal{E}^X_{\underline{\square}} \to \wedge^p_{\underline{\square}}$ *of contravariant functors :* $\underline{\square} \to \mathsf{Set}$. *We denote* $\omega = \{\omega_{\underline{n}}\,;\, n \geqq 0\}$, *where* $\omega_{\underline{n}} : \mathcal{E}^X(\square^n) \to \wedge^p(\square^n)$. *The set of cubical differential forms on X is denoted by* $\mathcal{A}^p_{\underline{\square}}(X)$ *and* $\mathcal{A}^*_{\underline{\square}}(X) = \bigoplus_p \mathcal{A}^p_{\underline{\square}}(X)$.

We denote by $\square^* : \mathcal{A}^p_{\mathcal{C}}(X) \to \mathcal{A}^p_{\underline{\square}}(X)$ the natural map induced from $\square : \underline{\square} \to \mathsf{Convex}$.

Theorem 4.2. *The map* $\square^* : \mathcal{A}^p_{\mathcal{C}}(X) \to \mathcal{A}^p_{\underline{\square}}(X)$ *is monic.*

Proof: Assume that $\omega \in \mathcal{A}^p_{\mathcal{C}}(X)$ satisfies $\square^*(\omega) = 0 : \mathcal{E}^X_{\underline{\square}} \to \wedge^p_{\underline{\square}}$.

By induction on n, we show $\omega_A = 0$ for any convex n-domain A.

$(n = 0)$ In this case, we have $\mathcal{A}^0_{\mathcal{C}}(X) = \mathcal{A}^0_{\underline{\square}}(X)$ and $\omega_{points} = 0$.

$(n > 0)$ Let $P : A \to X$ be a plot of X, where A is a convex n-domain. For any element $u \in \mathrm{Int}\,A$, there is a small simplex $\square^n_u \subset \mathrm{Int}\,A$ such that $\mathrm{Int}\,\square^n_u \ni u$. Then there is a linear diffeomorphism $\phi : \square^n \approx \square^n_u$. Hence $P\circ\phi \in C^\infty_{\mathcal{C}}(\square^n, X)$ and we obtain

$$0 = \square^*(\omega)_n(P\circ\phi) = \omega_{\square^n}(P\circ\phi) = \phi^*(\omega_{\square^n_u}(P|_{\square^n_u})) = \phi^*(\omega_A(P)|_{\square^n_u}).$$

Since ϕ is a diffeomorphism, we have $\omega_A(P)|_{\square^n_u} = 0$ for any $u \in \operatorname{Int} A$. Thus we obtain $\omega_A(P) = 0$ on $\operatorname{Int} A$. Since $\omega_A(P)$ is continuous, $\omega_A(P) = 0$ on A. □

A differentiable map induces a homomorphism of cubical differential forms as follows:

Definition 4.3. *Let $f : X \to Y$ be a differentiable map between differentiable spaces $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$.*

(1) We obtain a homomorphism $f^\sharp : \mathcal{A}^p_\square(Y) \to \mathcal{A}^p_\square(X)$: let $\omega \in \mathcal{A}^p_\square(Y)$. Then

$$(f^\sharp\omega_{\underline{n}})(P) = \omega_{\underline{n}}(f\circ P) \quad \text{for any} \quad P \in \mathcal{E}^X_\square(\underline{n}),\ n \geq 0.$$

(2) If a differentiable map f is proper, then we have $f^\sharp(\mathcal{A}^p_{\square_c}(Y)) \subset \mathcal{A}^p_{\square_c}(X)$ by taking $K_{f^\sharp\omega} = f^{-1}(K_\omega)$ for any $\omega \in \mathcal{A}^p_{\square_c}(Y)$.

Definition 4.4 (External derivative)**.** *Let $X = (X, \mathcal{E})$ be a differentiable space. The external derivative $d : \mathcal{A}^p_\square(X) \to \mathcal{A}^{p+1}_\square(X)$ is defined as follows.*

$$(d\omega)_{\underline{n}}(P) = d(\omega_{\underline{n}}(P)) \quad \text{for an } n\text{-plot } P \in \mathcal{E}_\square(\underline{n}) = \mathcal{E}(\square^n).$$

Definition 4.5. *Let $X = (X, \mathcal{E})$ be a differentiable space.*

Cubical de Rham cohomology: $H^p_\square(X) = \dfrac{Z^p_\square(X)}{B^p_\square(X)}$,

where $Z^p_\square(X) = \operatorname{Ker} d \cap \mathcal{A}^p_\square(X)$ and $B^p_\square(X) = d(\mathcal{A}^p_\square(X))$.

Cubical de Rham cohomology with compact support: $H^p_{\square_c}(X) = \dfrac{Z^p_{\square_c}(X)}{B^p_{\square_c}(X)}$,

where $Z^p_{\square_c}(X) = \operatorname{Ker} d \cap \mathcal{A}^p_{\square_c}(X)$ and $B^p_{\square_c}(X) = d(\mathcal{A}^p_{\square_c}(X))$.

Example 4.6. *Let $X = (X, \mathcal{E}^X)$ be a differentiable space with $X = \{*\}$ one-point-set. Then we have $H^p_\square(\{*\}) = \mathbb{R}$ if $p = 0$ and 0 otherwise.*

Proposition 4.7. *Let $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$ be differentiable spaces.*

(1) For a differentiable map $f : X \to Y$, the homomorphism $f^\sharp : \mathcal{A}^*_\square(Y) \to \mathcal{A}^*_\square(X)$ induces a homomorphism $H^*_\square(Y) \to H^*_\square(X)$.

(2) If a differentiable map $f : X \to Y$ is proper, then the homomorphism $f^\sharp : \mathcal{A}^*_{\square_c}(Y) \to \mathcal{A}^*_{\square_c}(X)$ induces a homomorphism $f^* : H^*_{\square_c}(Y) \to H^*_{\square_c}(X)$.

Theorem 4.8. *By definition, we clearly have $H^*_\square(\coprod_\alpha X_\alpha) = \prod_\alpha H^*_\square(X_\alpha)$.*

Theorem 4.9. *$H^*_\square$ is a contravariant functor from* Differentiable *to* GradedAlgebra*.*

5. Homotopy invariance of cubical de Rham cohomology

Let $f_0, f_1 : X \to Y$ be homotopic differentiable maps between differentiable spaces $X = (X, \mathcal{E}^X)$ and $Y = (Y, \mathcal{E}^Y)$. Then there is a plot $f : I \to C^\infty_{\mathcal{C}}(X, Y)$ with $f(t) = f_t$ for $t = 0, 1$. In particular, for any n-plot $P : \square^n \to X$, $f{\cdot}P : \square^{n+1} = I{\times}\square^n \xrightarrow{f\cdot P} Y$ is an $n{+}1$-plot. Then, we obtain a homomorphism $D_f : \mathcal{A}^p_{\square}(Y) \to \mathcal{A}^{p-1}_{\square}(X)$ as follows: for any cubical differential p-form $\omega : \mathcal{E}^Y_{\square} \to \wedge^p_{\square}$ on Y, a $p{-}1$-form $D_f(\omega) : \mathcal{E}^X_{\square} \to \wedge^{p-1}_{\square}$ on X is defined by the following formula.

$$D_f(\omega)_{\underline{n}}(P) = \int_I \omega_{\underline{n+1}}(f{\cdot}P) : \square^n \to \wedge^{p-1}(T^*_n),$$

$$\left[\int_I \omega_{\underline{n+1}}(f{\cdot}P)\right](\boldsymbol{x}) = \sum_{i_2,\cdots,i_p} \int_0^1 a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt{\cdot}\, dx_{i_2} \wedge \cdots \wedge dx_{i_p},$$

where we assume $\omega_{\underline{n+1}}(f{\cdot}P) = \sum_{i_2,\cdots,i_p} a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(t,\boldsymbol{x})$ $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I{\times}\square^n = \square^{n+1}$ and $T^*_{n+1} = \mathbb{R}\, dt \oplus \bigoplus_{i=1}^{n} \mathbb{R}\, dx_i$.

Lemma 5.1. *For any ω, we obtain $dD(\omega)_{\underline{n}} + D(d\omega)_{\underline{n}} = f_1^\sharp\omega_{\underline{n}} - f_0^\sharp\omega_{\underline{n}}$. Thus, if $d\omega = 0$, then $f_0^\sharp\omega$ is cohomologous to $f_1^\sharp\omega$.*

Proof: First, let $\omega_{\underline{n+1}}(f{\cdot}P) = \sum_{i_2,\cdots,i_p} a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(t,\boldsymbol{x})$ $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Let $\mathrm{in}_t : \square^n \to I{\times}\square^n$ be the inclusion defined by $\mathrm{in}_t(\boldsymbol{x}) = (t,\boldsymbol{x})$ for $t = 0, 1$. Since $(f{\cdot}P)\circ\mathrm{in}_t = f_t\circ P$ for $t = 0,1$, we have $(f_t^\sharp\omega_{\underline{n}})(P) = \omega_{\underline{n}}(f_t\circ P) = \omega_{\underline{n}}((f{\cdot}P)\circ\mathrm{in}_t)$ $= \mathrm{in}_t^*\,\omega_{\underline{n+1}}(f{\cdot}P) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(t,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for $t = 0, 1$, $\boldsymbol{x} \in \square^n$.

Second, by definition, we have $d\omega_{\underline{n+1}}(f{\cdot}P) = \sum_i \sum_{i_2,\cdots,i_p} \frac{\partial a_{i_2,\cdots,i_p}}{\partial x_i}(t,\boldsymbol{x})\, dx_i \wedge dt \wedge dx_{i_2} \wedge \cdots \wedge$ $dx_{i_p} + \sum_{i_1,\cdots,i_p} \frac{\partial b_{i_1,\cdots,i_p}}{\partial t}(t,\boldsymbol{x})\, dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_i \sum_{i_1,\cdots,i_p} \frac{\partial b_{i_1,\cdots,i_p}}{\partial x_i}(t,\boldsymbol{x})\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, and hence we obtain $D(d\omega)_{\underline{n}}(P) = -\sum_i \sum_{i_2,\cdots,i_p} \int_I \frac{\partial a_{i_2,\cdots,i_p}}{\partial x_i}(t,\boldsymbol{x})\, dt{\cdot}\, dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} +$ $\sum_{i_1,\cdots,i_p} \int_I \frac{\partial b_{i_1,\cdots,i_p}}{\partial t}(t,\boldsymbol{x})\, dt{\cdot}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I{\times}\square^n$.

Third, we have $D_f(\omega)_{\underline{n}}(P) = \sum_{i_2,\cdots,i_p} \int_I a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt{\cdot}\, dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, and hence we obtain $dD_f(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2,\cdots,i_p} \int_I \frac{\partial a_{i_2,\cdots,i_p}}{\partial x_i}(t,\boldsymbol{x})\, dt{\cdot}\, dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I{\times}\square^n$.

Hence $[dD_f(\omega)_{\underline{n}}(P) + D_f(d\omega)_{\underline{n}}(P)](\boldsymbol{x}) = \sum_{i_1,\cdots,i_p} \int_I \frac{\partial b_{i_1,\cdots,i_p}}{\partial t}(t,\boldsymbol{x})\, dt{\cdot}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} =$ $\sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(1,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} - \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(0,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $\boldsymbol{x} \in \square^n$. Thus we obtain $dD_f(\omega)_{\underline{n}}(P) + D_f(d\omega)_{\underline{n}}(P) = (f_1^\sharp\omega_{\underline{n}})(P) - (f_0^\sharp\omega_{\underline{n}})(P)$, which implies the lemma. □

It immediately implies the following theorem.

Theorem 5.2. *If two differentiable maps $f_0,\ f_1 : X \to Y$ between differentiable spaces are homotopic in $C^\infty_{\mathcal{C}}(X,Y)$, then they induce the same homomorphism*

$$f_0^* = f_1^* : H^*_\square(Y) \to H^*_\square(X).$$

6. Hurewicz homomorphism

First, we give a definition of paths and fundamental groupoid of a differentiable space.

Definition 6.1. *In this paper, a path from $a \in X$ to $b \in X$ in a differentiable space X means a differentiable map $\ell : I \to X$ such that $\ell(0) = a$ and $\ell(1) = b$. We denote by $\pi_0(X)$ the set of path-connected components of X, as usual.*

Definition 6.2. *Let* Cat *be the category of all small categories. The fundamental groupoid functor $\underline{\pi}_1 : \mathsf{Differentiable} \to \mathsf{Cat}$ is as follows:*

(1) For a differentiable space X, the small category $\underline{\pi}_1(X)$ is defined by $\mathrm{Obj}(\underline{\pi}_1(X)) = X$ and $\mathrm{Hom}_{\underline{\pi}_1(X)}(x_0,x_1)$ is the set of homotopy classes of all differentiable maps $\ell : I \to X$ with $\ell(0) = x_0$ and $\ell(1) = x_1$ for any $x_0, x_1 \in X$.

(2) For a differentitable map $f : Y \to X$, the functor $f_* : \underline{\pi}_1(Y) \to \underline{\pi}_1(X)$ is defined by $f_* = f : Y \to X$ and $f_*([\ell]) = [f\circ\ell]$ for any $[\ell] \in \underline{\pi}_1(Y)$.

Definition 6.3. *The functor $\underline{\mathbb{R}} : \mathsf{Differentiable} \to \mathsf{Cat}$ is defined as follows:*

(1) For a differentiable space X, the small category $\underline{\mathbb{R}}(X)$ is defined by $\mathrm{Obj}(\underline{\mathbb{R}}(X)) = X$ and $\mathrm{Hom}_{\underline{\mathbb{R}}(X)}(x_0,x_1) = \mathbb{R}$ for any $x_0, x_1 \in X$, and the composition is given by addition of real numbers.

(2) For a differentitable map $f : Y \to X$, the functor $f_* : \underline{\mathbb{R}}(Y) \to \underline{\mathbb{R}}(X)$ is defined by $f_* = f : Y \to X$ and $f_* = \mathrm{id} : \mathbb{R} \to \mathbb{R}$.

Definition 6.4. *The Hurewicz homomorphism $\rho : Z^1_\square(X) \to \mathrm{Hom}(\underline{\pi}_1(X), \underline{\mathbb{R}}(X))$ (the set of functors) is defined for any $\omega \in Z^1_\square(X)$ by $\rho(\omega)(x) = x$ for any $x \in \mathrm{Obj}(\underline{\pi}_1(X)) = X$ and $\rho(\omega)([\ell]) = \displaystyle\int_I \omega_{\underline{1}}(\ell)$ for any $[\ell] \in \mathrm{Hom}_{\underline{\pi}_1(X)}$, which is natural, in other words, the diagram below is commutative for any differentiable map $f : Y \to X$ between differentiable spaces.*

$$\begin{array}{ccc} Z^1_{\underline{\square}}(X) & \xrightarrow{\rho} & \mathrm{Hom}(\underline{\pi}_1(X), \underline{\mathbb{R}}(X)) \\ {\scriptstyle f^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}(f_*,\mathrm{id})} \\ Z^1_{\underline{\square}}(Y) & \xrightarrow{\rho} & \mathrm{Hom}(\underline{\pi}_1(Y), \underline{\mathbb{R}}(Y)) \end{array}$$

(well-defined) Let $\ell_0 \sim \ell_1$ with $\ell_t(\epsilon) = x_\epsilon \in X$, $t = 0, 1$ and $\epsilon = 0, 1$. Then there is a 2-plot $\hat{\ell} : \square^2 \to X$ such that $\hat{\ell}(\epsilon, s) = \ell_\epsilon(s)$ and $\hat{\ell}(t, \epsilon) = x_\epsilon$ for $\epsilon = 0, 1$. Hence we have $\hat{\ell} \circ \partial^\epsilon_1 = \ell_\epsilon$, $\epsilon = 0, 1$ and $\hat{\ell} \circ \partial^\epsilon_2 = c_{x_\epsilon} = c_{x_\epsilon} \circ \varepsilon_1$. Let $\omega_{\underline{2}}(\hat{\ell}) = a(t, s)\, dt + b(t, s)\, ds \in \wedge^1(\square^2)$. Then we have $\omega_{\underline{2}}(\ell_\epsilon) = \omega_{\underline{2}}(\hat{\ell} \circ \partial^\epsilon_1) = \partial^{\epsilon *}_1 \omega_{\underline{2}}(\hat{\ell}) = b(\epsilon, s)\, ds$, $\epsilon = 0, 1$. Similarly, $0 = \varepsilon^*_1 \omega_*(c_{x_\epsilon}) = \omega_{\underline{2}}(c_{x_\epsilon} \circ \varepsilon_1) = \omega_{\underline{2}}(\hat{\ell} \circ \partial^\epsilon_2) = \partial^{\epsilon *}_2 \omega_{\underline{2}}(\hat{\ell}) = a(t, \epsilon)\, dt$ which implies $a(t, \epsilon) = 0$, $\epsilon = 0, 1$. On the other hand by Green's formula, we obtain that $\displaystyle\int_{\partial \square^2} (\omega_{\underline{2}}(\hat{\ell})|_{\partial \square^2}) = \int_{\square^2} d\omega = 0$, since ω is a closed form. Then it follows that $\displaystyle\int_{\{1\} \times I} (\omega_{\underline{2}}(\hat{\ell})|_{\{1\} \times I}) - \int_{\{0\} \times I} (\omega_{\underline{2}}(\hat{\ell})|_{\{0\} \times I}) = 0$, and hence $\displaystyle\int_I \omega_{\underline{1}}(\ell_1) = \int_I \omega_{\underline{1}}(\ell_0)$, and ρ is well-defined. The additivity of ρ is clear by definition.

(naturality) Let $f : Y \to X$ be a differentiable map. Then f induces both $f^* : Z^1_{\underline{\square}}(X) \to Z^1_{\underline{\square}}(Y)$ and $f_* : \underline{\pi}_1(Y) \to \underline{\pi}_1(X)$. The latter homomorphism induces

$$\mathrm{Hom}(f_*, \mathrm{id}) : \mathrm{Hom}(\underline{\pi}_1(X), \underline{\mathbb{R}}(X)) \to \mathrm{Hom}(\underline{\pi}_1(Y), \underline{\mathbb{R}}(Y)).$$

Then, for any $\omega \in Z^1_{\underline{\square}}(X)$ and $[\ell] \in \underline{\pi}_1(X)$, it follows that

$$\rho(f^*(\omega))([\ell]) = \int_I (f^\sharp \omega_{\underline{1}})(\ell) = \int_I \omega_{\underline{1}}(f \circ \ell) = \rho(\omega)([f \circ \ell]) = \rho(\omega) \circ f_*([\ell])$$

and hence we have $\rho \circ f^* = \mathrm{Hom}(f_*, \mathrm{id}) \circ \rho$ which implies the naturality of ρ.

Definition 6.5. *For any differentiable space X, we define a groupoid $\underline{X}$ in which the set of objects is equal to $X = \mathrm{Obj}(\underline{\pi}_1(X))$, and the set of morphisms is obtained from $\mathrm{Hom}_{\underline{\pi}_1(X)}$ by identifying all the morphisms which have starting and ending objects in common.*

Then there clearly is a natural projection $\mathrm{pr} : \underline{\pi}_1(X) \to \underline{X}$ inducing a monomorphism $\mathrm{pr}^* : \mathrm{Hom}(\underline{X}, \underline{\mathbb{R}}(X)) \hookrightarrow \mathrm{Hom}(\underline{\pi}_1(X), \underline{\mathbb{R}}(X))$.

Definition 6.6. *We denote the cokernel of pr^* by $\mathrm{Hom}(\underline{\pi}_1(X), \mathbb{R})$.*

If $\omega = d\phi$ for some $\phi \in \mathcal{A}^0_{\underline{\square}}(X)$, then, for any path ℓ from x_0 to x_1, we have $\rho(\omega)([\ell]) = \rho(d\phi)([\ell]) = \displaystyle\int_I d(\phi_I)(\ell) = [\phi_I(\ell)(t)]^{t=1}_{t=0} = \phi_I(\ell)(1) - \phi_I(\ell)(0)$, by the fundamental theorem of calculus. Hence $\phi_I(\ell)(\epsilon) = \phi_I(\ell)(\partial^\epsilon_1(*)) = \partial^{\epsilon *}_1(\phi_I(\ell))(*) = \phi_{\{*\}}(\ell \circ \partial^\epsilon_1)(*) =$

$\phi_{\{*\}}(\ell(\epsilon))(*) = \phi_{\{*\}}(c_{x_\epsilon})(*)$ is depending only on x_ϵ the starting and ending objects of $[\ell] \in \underline{\pi}_1(X)$. Thus the functor $\rho(\omega) : \underline{\pi}_1(X) \to \underline{\mathbb{R}}(X)$ induces a functor $\Phi(\omega) : \underline{X} \to \underline{\mathbb{R}}(X)$ such that $\rho(\omega) = \Phi(\omega)\circ\mathrm{pr}$, in other words, $\rho(B^1_{\underline{\square}}(X))$ is in the image of pr^*. Thus ρ induces a homomorphism $\rho_* : H^*_{\underline{\square}}(X) \to \mathrm{Hom}(\underline{\pi}_1(X), \mathbb{R})$.

Theorem 6.7. $\rho_* : H^1_{\underline{\square}}(X) \to \mathrm{Hom}(\underline{\pi}_1(X), \mathbb{R})$ *is a monomorphism.*

Proof: Assume that $\rho_*([\omega]) = 0$. Then we have $\rho(\omega) \in \mathrm{Im}\,\mathrm{pr}^*$. Thus there is a functor $\Phi(\omega) : \underline{X} \to \underline{\mathbb{R}}$ such that $\rho(\omega) = \Phi(\omega)\circ\mathrm{pr}$. Let $\{x_\alpha\,;\, \alpha \in \pi_0(X)\}$ be a complete set of representatives of $\pi_0(X)$. For any $P \in \mathcal{E}(\square^n)$, a map $F(P) : \square^n \to \mathbb{R}$ is given by

$$F(P)(\boldsymbol{x}) = \int_I \omega_{\underline{1}}(\ell_x) + \int_I \gamma^*_{\boldsymbol{x}}\omega_{\underline{n}}(\square^n),\ x = P(\mathbf{0}),$$

where ℓ_x is a path from x_α, $\alpha = [x] \in \pi_0(X)$, to x in X and γ is a path from $\mathbf{0}$ to $\boldsymbol{x}$ in $\square^n$. Then $F(P) : \square^n \to \wedge^0$ is well-defined smooth map by the equality $\displaystyle\int_I \omega_{\underline{1}}(\ell_x) = \rho(\omega)([\ell_x]) = \Phi(\omega)(\mathrm{pr}([\ell_x]))$ which is not depending on the choice of ℓ_x, and hence it gives a 0-form $F : \mathcal{E}(\square^n) \to \wedge^0(\square^n)$ so that $dF = \omega$. Thus $[\omega] = 0$ and ρ_* is a monomorphism. □

7. Partition of unity

Let X be a differentiable space. In this section, we assume that there are subsets $A, B \subset X$ such that $\mathcal{U} = \{\mathrm{Int}\,A, \mathrm{Int}\,B\}$ gives an open covering of X.

Definition 7.1. *A pair* (ρ^A, ρ^B) *of differentiable* 0*-forms* ρ^A *and* ρ^B *is called a partition of unity belonging to an open covering* $\mathcal{U}$ *of* X*, if, for any plot* $P : \square^n \to X$*,* $\mathrm{Supp}\,\rho^A_{\underline{n}}(P) \subset P^{-1}(\mathrm{Int}\,A)$*,* $\mathrm{Supp}\,\rho^B_{\underline{n}}(P) \subset P^{-1}(\mathrm{Int}\,B)$ *and* $\rho^A_{\underline{n}}(P) + \rho^B_{\underline{n}}(P) = 1$ *on* $\square^n$*.*

To obtain a well-defined smooth function by extending or gluing smooth functions on cubic sets, we use a fixed smooth stabilizer function $\hat{\lambda} : \mathbb{R} \to I$ (see [7]) which satisfies

(1) $\hat{\lambda}(-t) = 0$, $\hat{\lambda}(1+t) = 1$, $t \geq 0$ and (2) $\hat{\lambda}$ is strictly increasing on $I = [0, 1]$.

Using $\hat{\lambda}$, we define a smooth function $\lambda_{a,b} : I \to I$, for any $a, b \in \mathbb{R}$ with $a<b$, by

$$\lambda_{a,b}(t) = \hat{\lambda}\left(\tfrac{t-a-\epsilon}{b-a-2\epsilon}\right)$$

for a small $\epsilon>0$ enough to satisfy $\frac{b-a}{2}>\epsilon>0$.

Using it, we show the existence of a partition of unity as follows.

Theorem 7.2. *Let* X *be a differentiable space with an open covering* $\{\mathrm{Int}\,A, \mathrm{Int}\,B\}$*,* $A, B \subset X$*. Then there exists a partition of unity* $\boldsymbol{\rho} = \{\rho^A, \rho^B\}$ *belonging to* $\{\mathrm{Int}\,A, \mathrm{Int}\,B\}$*. If the underlying topology on* X *is normal,* $\boldsymbol{\rho}$ *can be chosen as normal, in other words, there are* *closed sets* G_A*,* G_B *in* X *such that* $X \smallsetminus \operatorname{Int} B \subset G_A \subset \operatorname{Int} A$*,* $X \smallsetminus \operatorname{Int} A \subset G_B \subset \operatorname{Int} B$ *and* $\operatorname{Supp} \rho^A_{\underline{n}}(P) \subset P^{-1}(G_A)$ *and* $\operatorname{Supp} \rho^B_{\underline{n}}(P) \subset P^{-1}(G_B)$ *for all* $n \geq 0$ *and* $P \in \mathcal{E}^X(\square^n)$*.*

The above theorem implies the exactness of Mayer-Vietoris exact sequence as follows.

Corollary 7.3. *Let* X *be a differentiable space with an open covering* $\mathcal{U} = \{\operatorname{Int} A, \operatorname{Int} B\}$*,* $A, B \subset X$*. Then we have the following long exact sequence.*

$$\cdots \to H^q_{\underline{\square}}(X) \to H^q_{\underline{\square}}(A) \oplus H^q_{\underline{\square}}(B) \to H^q_{\underline{\square}}(A \cap B)$$
$$\to H^{q+1}_{\underline{\square}}(X) \to H^{q+1}_{\underline{\square}}(A) \oplus H^{q+1}_{\underline{\square}}(B) \to H^{q+1}_{\underline{\square}}(A \cap B) \cdots$$

Proof of Theorem 7.2. If X is normal, there is a continuous function $\rho : X \to I$ with $X \smallsetminus \operatorname{Int} B \subset \rho^{-1}(0)$ and $X \smallsetminus \operatorname{Int} A \subset \rho^{-1}(1)$. Otherwise, we define a function $\rho : X \to I$ by

$$\rho(x) = \begin{cases} 1, & x \in \operatorname{Int} A \smallsetminus \operatorname{Int} B, \\ 1/2, & x \in \operatorname{Int} A \cap \operatorname{Int} B, \\ 0 & x \in \operatorname{Int} B \smallsetminus \operatorname{Int} A. \end{cases}$$

Let $G_A = \rho^{-1}([0, \frac{2}{3}]) \subset X \smallsetminus \rho^{-1}(1) \subset \operatorname{Int} A$ and $G_B = \rho^{-1}([\frac{1}{3}, 1]) \subset X \smallsetminus \rho^{-1}(0) \subset \operatorname{Int} B$. Then $\operatorname{Int} G_A \cup \operatorname{Int} G_B \supset \rho^{-1}([0, \frac{2}{3})) \cup \rho^{-1}((\frac{1}{3}, 1]) = \rho^{-1}([0, \frac{2}{3}) \cup (\frac{1}{3}, 1]) = X$. Thus it is sufficient to construct a partition of unity $\{\rho^A, \rho^B\}$ belonging to $\mathcal{U} = \{\operatorname{Int} G_A, \operatorname{Int} G_B\}$: by induction on n, we construct functions $\rho^A_{\underline{n}}(P)$, $\rho^B_{\underline{n}}(P) : \square^n \to I$ for any n-plot $P : \square^n \to X$, with conditions (1) through (4) below for $F = A, B$ and $\epsilon = 0, 1$.

(1) a) $\rho^F_{\underline{n}}(P \circ \varepsilon_i) = \rho^F_{\underline{n-1}}(P) \circ \varepsilon_i$, $1 \leq i \leq n+1$, b) $\rho^F_{\underline{n-1}}(P \circ \partial^\epsilon_i) = \rho^F_{\underline{n}}(P) \circ \partial^\epsilon_i$, $1 \leq i \leq n$,

(2) $\rho^A_{\underline{n}}(P) + \rho^B_{\underline{n}}(P) = 1 : \square^n \to \mathbb{R}$, (3) $\operatorname{Supp} \rho^F_{\underline{n}}(P) \subset P^{-1}(\operatorname{Int} G_F) \subset \square^n$,

(4) $\rho_F(P) \circ \partial^{1-t}_i = \rho_F(P) \circ \partial^1_i$ and $\rho_F(P) \circ \partial^t_i = \rho_F(P) \circ \partial^0_i$ for all $0 \leq t \leq a$ for sufficiently small $a > 0$, where ∂^t_i is defined by $\partial^t_i(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_{n-1})$.

($n = 0$) For any plot $P : \square^0 = \{*\} \to X$, we define $\rho^A_{\underline{n}}(P) = \rho(P(*))$ and $\rho^B_{\underline{n}}(P) = 1 - \rho^A_{\underline{n}}(P)$, which satisfy (2) and (3), though (1) and (4) are empty conditions in this case.

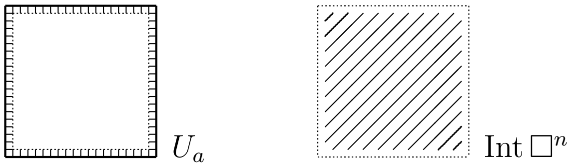
($n > 0$) We may assume a plot $P : \square^n \to X$ is non-degenerate by (1) a).

Firstly, $P^{-1}\mathcal{U} = \{P^{-1}(\operatorname{Int} A), P^{-1}(\operatorname{Int} B)\}$ is an open covering of $\square^n \subset \mathbb{R}^n$, and hence we have a partition of unity $\{\varphi^A, \varphi^B\}$ belonging to $P^{-1}\mathcal{U}$ on $\square^n$.

Secondly, by the induction hypothesis, there is a small $a > 0$ for the condition (4). Let U_a be the a-neighbourhood of $\partial \square^n$. For $F = A, B$, we define $\hat{\rho}^F_{\underline{n}}(P) : U_a \to \mathbb{R}$ by

$$\hat{\rho}^F_{\underline{n}}(P) \circ \partial^{\epsilon \pm t}_i = \rho^F_{\underline{n-1}}(P \circ \partial^\epsilon_i), \ 0 \leq t < a, \ 1 \leq i \leq n, \ \epsilon = 0, 1,$$

where we denote $\epsilon \pm t = \epsilon + (-1)^\epsilon t$, and then we obtain $\operatorname{Supp} \hat{\rho}^F_{\underline{n}}(P) \subset P^{-1}(\operatorname{Int} G_F) \cap U_a$, if we choose $a > 0$ small enough.



Thirdly, since two open sets U_a and $\mathrm{Int}\,\square^n$ form an open covering of $\square^n$, we also have a partition of unity $(\psi_\partial, \psi_\circ)$ belonging to $\{U_a, \mathrm{Int}\,\square^n\}$ given by $\psi_\partial = (\lambda_{1-a,1})^n$ and $\psi_\circ = 1 - \psi_\partial$ so that we have $\mathrm{Supp}\,\psi_\partial \subset U_a$ and $\mathrm{Supp}\,\psi_\circ \subset \mathrm{Int}\,\square^n$. Then, for $F = A, B$, $\psi_\partial|_{U_a}\cdot\hat{\rho}^F_{\underline{n}}(P)$ is defined on U_a with value 0 on $U_a \smallsetminus \mathrm{Supp}\,\psi_\partial$. Hence by filling 0 outside $\mathrm{Supp}\,\psi_\partial$, we obtain a smooth map $\widehat{\psi_\partial\rho^F_{\underline{n}}} : \square^n \to \mathbb{R}$ on entire $\square^n$, as the 0-extension of $\psi_\partial|_{U_a}\cdot\hat{\rho}^F_{\underline{n}}(P) : U_a \to \mathbb{R}$.

Finally, let $\rho^F_{\underline{n}}(P) = \widehat{\psi_\partial\rho^F_{\underline{n}}} + \psi_\circ\cdot\varphi^F$ for $F = A, B$. Then $\mathrm{Supp}\,\rho^F_{\underline{n}}(P) \subset \mathrm{Supp}\,\widehat{\psi_\partial\rho^F_{\underline{n}}} \cup \mathrm{Supp}(\psi_\circ\cdot\varphi^F) \subset (\mathrm{Supp}\,\psi_\partial \cap \mathrm{Supp}\,\hat{\rho}^F_{\underline{n}}) \cup (\mathrm{Supp}\,\psi_\circ \cap \mathrm{Supp}\,\varphi^F) \subset (U_a \cap P^{-1}(\mathrm{Int}\,G_F)) \cup (\mathrm{Int}\,\square^n \cap P^{-1}(\mathrm{Int}\,G_F)) = P^{-1}(\mathrm{Int}\,G_F)$. By definition, we also have

$$\rho^A_{\underline{n}}(P) + \rho^B_{\underline{n}}(P) = \widehat{\psi_\partial\rho^A_{\underline{n}}} + \widehat{\psi_\partial\rho^B_{\underline{n}}} + \psi_\circ\cdot\varphi^A + \psi_\circ\cdot\varphi^B = \psi_\partial + \psi_\circ = 1 \quad \text{on} \quad \square^n,$$

which implies that $(\rho^A_{\underline{n}}(P), \rho^B_{\underline{n}}(P))$ gives a partition of unity belonging to the open covering $\{P^{-1}(\mathrm{Int}\,A), P^{-1}(\mathrm{Int}\,B)\}$ of $\square^n$. By definition, $(\rho^A_{\underline{n}}(P), \rho^B_{\underline{n}}(P))$ satisfies the conditions (1) through (4), and it completes the induction step. The latter part is clear. □

8. Excision theorem

Let $X = (X, \mathcal{E}^X)$ be a differentiable space and $\mathcal{U}$ an open covering of X. We denote $\mathcal{E}^{\mathcal{U}} = \{P \in \mathcal{E}^X ; \mathrm{Im}\,P \subset U \text{ for some } U \in \mathcal{U}\}$. Then we regard $\mathcal{E}^{\mathcal{U}}$ as a functor $\mathcal{E}^{\mathcal{U}} : \mathsf{Convex} \to \mathsf{Set}$ which is given by $\mathcal{E}^{\mathcal{U}}(C) = \{P \in \mathcal{E}^{\mathcal{U}},\ \mathrm{Dom}\,P = C\}$ for $C \in \mathrm{Obj}(\mathsf{Convex})$ and $\mathcal{E}^{\mathcal{U}}(f) = \mathcal{E}^X(f)|_{\mathcal{E}^{\mathcal{U}}(C)} : \mathcal{E}^{\mathcal{U}}(C) \to \mathcal{E}^{\mathcal{U}}(C')$ for a smooth map $f : C' \to C$ in Convex. When $\mathcal{U} = \{X\}$, we have $\mathcal{E}^{\{X\}} = \mathcal{E}^X$. We also denote $\mathcal{E}^{\mathcal{U}}_{\underline{\square}} = \mathcal{E}^{\mathcal{U}}\circ\underline{\square} : \underline{\square} \to \mathsf{Set}$.

Definition 8.1. *A natural transformation $\omega : \mathcal{E}^{\mathcal{U}}_{\underline{\square}} \to \wedge^p_{\underline{\square}}$ is called a cubical differencial p-form w.r.t. an open covering $\mathcal{U}$ of X. $\mathcal{A}^p_{\underline{\square}}(\mathcal{U})$ denotes the set of all cubical differential p-form w.r.t. an open covering $\mathcal{U}$ of X. For example, $\mathcal{A}^p_{\underline{\square}}(\{X\}) = \mathcal{A}^p_{\underline{\square}}(X)$.*

We introduce a notion of a q-cubic set in $\mathbb{R}^n$ using induction on $q \geq -1$ up to n.

($q=-1$): The empty set $\emptyset$ is a -1-cubic set in $\mathbb{R}^n$.

($n \geq q \geq 0$): (1) if $\sigma \subset L$ is a $(q-1)$-cubic set in $\mathbb{R}^n$ and $\boldsymbol{b} \notin L$, where L is a hyperplane of dimension $q-1$ in $\mathbb{R}^n$, then $\sigma * \boldsymbol{b} = \{t\boldsymbol{x} + (1-t)\boldsymbol{b}\, ;\, \boldsymbol{x} \in \sigma,\ t \in I\}$ is a q-cubic set in $\mathbb{R}^n$ with faces τ and $\tau * \boldsymbol{b}$, where τ is a face of σ, including $\emptyset$ and $\emptyset * \boldsymbol{b} = \boldsymbol{b}$.

(2) if $\sigma \subset \mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{n-i}$ is a $(q-1)$-cubic set in $\mathbb{R}^n$ with $q\geq 1$, then the product set $\sigma\times I = \{(\boldsymbol{x}_{i-1}, t, \boldsymbol{x}'_{n-i})\,;\,(\boldsymbol{x}_{i-1}, 0, \boldsymbol{x}'_{n-i})\in\sigma,\ t\in I\}$ is a q-cubic set in $\mathbb{R}^n$ with faces $\tau\times\{0\}$, $\tau\times\{1\}$ and $\tau\times I$, where τ is a face of σ, including $\emptyset$.

We denote by $C(n)^q$ the set of q-cubic sets in $\mathbb{R}^n$ and $C(n) = \{\emptyset\} \cup \bigcup_{q\geq 0} C(n)^q$, $n \geq 0$. We denote $\tau < \sigma$ if $\tau \in C(n)$ is a face of $\sigma \in C(n)$ and denote $\partial\sigma = \bigcup_{\tau<\sigma} \sigma$. We fix a relative diffeomorphism $\phi_\sigma : (\square^q, \partial\square^q) \twoheadrightarrow (\sigma, \partial\sigma)$ for each q-cubic set σ in $\mathbb{R}^n$, $q\geq 0$.

A subset $K \subset C(n)$ is called a cubical complex if it satisfies the following conditions.

(1) $\emptyset \in K$, (2) $\tau < \sigma \ \&\ \sigma \in K \implies \tau \in K$,

(3) $\tau, \sigma \in K \implies \tau\cap\sigma \in K \ \&\ \tau\cap\sigma < \tau \ \&\ \tau\cap\sigma < \sigma$.

For any cubical complex $K\subset C(n)$, we denote $K^q = \{\sigma\in K\,;\,\sigma \text{ is a } q\text{-cubic set}\}$, $n\geq 0$ and $|K| = \bigcup_{\sigma\in K} \sigma$. For any cubical complexes K and L, a map $f : |L| \to |K|$ in Convex is called 'polyhedral' w.r.t. L and K, if $f(\sigma) \in K$ for any $\sigma \in L$. If a cubical complex $K\subset C(n)$ satisfies $|K| = \square^n$, we call K a 'cubical subdivision' of an n-cube $\square^n$.

Definition 8.2. *We define a category* $\mathsf{SubDiv}_{\mathcal{U}}$ *as follows:*

Object: $\mathrm{Obj}(\mathsf{SubDiv}_{\mathcal{U}}) = \{(K, P) \in C(n)\times\mathcal{E}^X(\square^n)\,;\,|K|=\square^n,\ \forall_{\sigma\in K}\, P|_\sigma \in\mathcal{E}^{\mathcal{U}},\ n\geq 0\}$,

Morphism: $\mathsf{SubDiv}_{\mathcal{U}}((L, Q), (K, P)) = \{f : |L| \subset |K| \textit{ polyhedral}\,;\, Q = P|_{|L|}\}$.

Let $\mathsf{SubDiv}_X = \mathsf{SubDiv}_{\{X\}}$. Then there is an embedding $\iota^{\mathcal{U}}_{SD} : \mathsf{SubDiv}_{\mathcal{U}} \hookrightarrow \mathsf{SubDiv}_X$.

Theorem 8.3. *There is a functor* $\mathrm{Sd}^*_{\mathcal{U}} : \mathsf{SubDiv}_X \twoheadrightarrow \mathsf{SubDiv}_{\mathcal{U}}$ *such that* $\mathrm{Sd}^*_{\mathcal{U}}\circ\iota^{\mathcal{U}}_{SD} = \mathrm{id}$.

Proof: We construct a functor $\mathrm{Sd}_{\mathcal{U}} : \mathsf{SubDiv}_X \to \mathsf{SubDiv}_X$ satisfying $\mathrm{Sd}_{\mathcal{U}}\circ\iota^{\mathcal{U}}_{SD} = \iota^{\mathcal{U}}_{SD}$.

Firstly, for $(K, P) \in \mathcal{O}(\mathsf{SubDiv}_X)$, $K \subset C(n)$ is a cubical subdivision of $|K| = \square^n = \mathrm{Dom}\,P$. Let $K_P(\mathcal{U}) = \{\sigma\in K\,;\,\exists_{U\in\mathcal{U}}\, P(\sigma)\subset U\} < K$. We define $\mathrm{Sd}_{\mathcal{U}}(K, P) = (\mathrm{Sd}^{\mathcal{U}}_P(K), P)$ by induction on dimension of a cubic set in K.

$$\mathrm{Sd}^{\mathcal{U}}_P(K)^0 = K^0 \cup \{\boldsymbol{b}_\sigma\,;\,\sigma \in K\smallsetminus K_P(\mathcal{U})\},$$
$$\mathrm{Sd}^{\mathcal{U}}_P(K)^q = K_P(\mathcal{U})^q \ \cup\ \{\rho{*}\boldsymbol{b}_\sigma\,;\,\rho \in \mathrm{Sd}^{\mathcal{U}}_P(\partial\sigma)^{q-1},\ \sigma \in K\smallsetminus K_P(\mathcal{U})\},$$

where $\partial\sigma$ denotes the subcomplex $\{\tau \in K\,;\,\tau<\sigma\}$ of K.

Secondly, for any map $f : (L, Q) \to (K, P)$, we have $L \subset K$ and $Q = P|_{|L|}$. Then by definition, we have $\mathrm{Sd}_{\mathcal{U}}(L)\subset\mathrm{Sd}_{\mathcal{U}}(K)$, and hence the inclusion $f : |\mathrm{Sd}_{\mathcal{U}}(L)| = |L| \subset |K| = |\mathrm{Sd}_{\mathcal{U}}(K)|$ is again polyhedral. Thus we obtain $\mathrm{Sd}_{\mathcal{U}}(f) = f : \mathrm{Sd}_{\mathcal{U}}(L, Q) \to \mathrm{Sd}_{\mathcal{U}}(K, P)$.

Thirdly, we give a distance of subcomplexes K and $K_P(\mathcal{U})$ defined as follows:

$$\varepsilon^{\mathcal{U}}_P(K) = \mathrm{Min}\,\{d(\tau, \boldsymbol{x}) \mid \tau \subset P^{-1}(U) \not\ni \boldsymbol{x},\ U \in \mathcal{U}\ \ \&\ \tau \text{ is maximal in } K_P(\mathcal{U})\}\,,$$
$$d^{\mathcal{U}}_P(K) = \mathrm{Max}\,\{d(\tau, \boldsymbol{x}) \mid \tau\cap\sigma \neq \emptyset,\ \boldsymbol{x} \in \sigma \in K\ \&\ \tau \text{ is maximal in } K_P(\mathcal{U})\}\,,$$

where $d(\tau, \boldsymbol{x})$ denotes the distance in $\square^n$ of τ and $\boldsymbol{x}$, and hence $\varepsilon^{\mathcal{U}}_P(K) > 0$. We can easily see that $d_{\mathcal{U}}(\mathrm{Sd}^{\mathcal{U}}_P(K)) \leq \frac{n}{n+1}\, d_{\mathcal{U}}(K)$ and hence that, for sufficiently large $r>0$, the r-times iteration of $\mathrm{Sd}^{\mathcal{U}}_P$ satisfies $d^{\mathcal{U}}_P((\mathrm{Sd}^{\mathcal{U}}_P)^r(K)) < \varepsilon^{\mathcal{U}}_P(K)$. Thus $\mathrm{Sd}^r_{\mathcal{U}}(K,P) \in \mathsf{SubDiv}_{\mathcal{U}}$.

Finally, when $(K,P) \in \mathsf{SubDiv}_{\mathcal{U}}$, we have $\mathrm{Sd}^{\mathcal{U}}_P(K,P) = (K,P)$ by definition, and hence $\mathrm{Sd}^*_{\mathcal{U}}$ the sufficiently many times iteration of $\mathrm{Sd}_{\mathcal{U}}$ on each (K,P) is a desired functor. □

Definition 8.4. *A functor* $\mathrm{Td}_{\mathcal{U}} : \mathsf{SubDiv}_X \to \mathsf{SubDiv}_X$ *given by* $\mathrm{Td}_{\mathcal{U}}(K,P) = (\mathrm{Td}^{\mathcal{U}}_P(K), \hat{P})$ *for* $(K,P) \in \mathrm{Obj}(\mathsf{SubDiv}_X)$ *is defined as follows: we denote* $\hat{P} = P \circ pr_1 : \square^n \times I \to X$ *which is a plot in* $\mathcal{E}^X(\square^{n+1})$*. Then a cubical subdivision* $\mathrm{Td}^{\mathcal{U}}_P(K)$ *of* $\square^{n+1}$ *is defined as follows:*

$$\begin{aligned}
\mathrm{Td}^{\mathcal{U}}_P(K)^0 &= K^0 \times \{0\} \cup \mathrm{Sd}^{\mathcal{U}}_P(K)^0 \times \{1\},\\
\mathrm{Td}^{\mathcal{U}}_P(K)^q &= K^q \times \{0\} \cup \mathrm{Sd}^{\mathcal{U}}_P(K)^q \times \{1\} \cup K_P(\mathcal{U})^{q-1} \times I\\
&\qquad \cup \{\rho * (\boldsymbol{b}_\sigma, 1)\,;\, \rho \in \mathrm{Td}^{\mathcal{U}}_P(\partial\sigma)^{q-1},\ \sigma \in K \smallsetminus K_P(\mathcal{U})\}.
\end{aligned}$$

Also for a map $f : (L,Q) \to (K,P)$*, we have* $L \subset K$ *and* $Q = P|_{|L|}$*. Then by definition, we have* $\mathrm{Td}_{\mathcal{U}}(L) \subset \mathrm{Td}_{\mathcal{U}}(K)$*, and hence the inclusion* $f \times \mathrm{id} : |\mathrm{Td}_{\mathcal{U}}(L)| = |L| \times I \subset |K| \times I = |\mathrm{Td}_{\mathcal{U}}(K)|$ *is again polyhedral. Thus we obtain* $\mathrm{Td}_{\mathcal{U}}(f) = f : \mathrm{Td}_{\mathcal{U}}(L,Q) \to \mathrm{Td}_{\mathcal{U}}(K,P)$*.*

Definition 8.5. *For any cubical differential p-form* $\omega \in \mathcal{A}^p_{\square}(\mathcal{U})$*, we have a cubical differential p-form* $\widetilde{\omega} \in \mathcal{A}^p_{\square}(\mathcal{U})$ *defined by* $\widetilde{\omega}_{\underline{n}}(P) = (\lambda^n)^* \omega_{\underline{n}}(P)$ *for any* $P \in \mathcal{E}^{\mathcal{U}}_{\underline{n}}$*,* $\lambda = \lambda_{0,1}$*. In addition, if* ω *is a differential p-form with compact support, then so is* $\widetilde{\omega}$*.*

Lemma 8.6. *There is a homomorphism* $D_{\mathcal{U}} : \mathcal{A}^*_{\square}(\mathcal{U}) \to \mathcal{A}^*_{\square}(\mathcal{U})$ *such that* $d D_{\mathcal{U}}(\omega)_{\underline{n}} + D_{\mathcal{U}}(d\omega)_{\underline{n}} = \widetilde{\omega}_{\underline{n}} - \omega_{\underline{n}}$ *and* $D_{\mathcal{U}}(\mathcal{A}^p_{\square_c}(\mathcal{U})) \subset \mathcal{A}^{p-1}_{\square_c}(\mathcal{U})$ *for any* $p \geq 0$*.*

Proof: Let $H : I \times I \to I$ be a smooth homotopy between $\mathrm{id} : I \to I$ and $\lambda : I \to I$, which gives rise to a smooth homotopy $H_n : \square^{n+1} = I \times \square^n \to \square^n$ of $\mathrm{id} : \square^n \to \square^n$ and $\lambda^n : \square^n \to \square^n$, $n \geq 0$. Then we have $H_n \circ \mathrm{in}_0 = \mathrm{id}$ and $H_n \circ \mathrm{in}_1 = \lambda^n$, where $\mathrm{in}_t : \square^n \hookrightarrow I \times \square^n$ is given by $\mathrm{in}_t(\boldsymbol{x}) = (t, \boldsymbol{x})$. For any cubical differential p-form $\omega : \mathcal{E}^{\mathcal{U}}_{\underline{\square}} \to \wedge^p_{\underline{\square}}$, a cubical $(p-1)$-form $D_{\mathcal{U}}(\omega) : \mathcal{E}^{\mathcal{U}}_{\underline{\square}} \to \wedge^{p-1}_{\underline{\square}}$ is defined on a plot $P \in \mathcal{E}^{\mathcal{U}}_{\underline{\square}}$, by the following formula.

$$D_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \int_I H^* \omega_{\underline{n}}(P) : \square^n \to \wedge^{p-1}(T^*_n),$$

$$\left[\int_I H^* \omega_{\underline{n}}(P)\right](\boldsymbol{x}) = \sum_{i_2,\cdots,i_p} \int_0^1 a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt \cdot dx_{i_2} \wedge \cdots \wedge dx_{i_p},$$

where we assume $H^*\omega_{\underline{n}}(P) = \sum_{i_2,\cdots,i_p} a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(t,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} : I \times \square^n \to \wedge^{p-1}(T^*_{n+1})$, $(t,\boldsymbol{x}) \in I \times \square^n$ and $T^*_{n+1} = \mathbb{R}\, dt \oplus \bigoplus_{i=1}^{n} \mathbb{R}\, dx_i$.

First, let $\mathrm{in}_t : \square^n \to I \times \square^n$ be the inclusion defined by $\mathrm{in}_t(\boldsymbol{x}) = (t, \boldsymbol{x})$ for $t = 0, 1$. By $H \circ \mathrm{in}_0 = \mathrm{id}$, we have $\omega_{\underline{n}}(P) = \mathrm{id}^* \omega_{\underline{n}}(P) = \mathrm{in}_0^* H^* \omega_{\underline{n}}(P) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(0,\boldsymbol{x})\, dx_{i_1} \wedge$

$\cdots \wedge dx_{i_p}$. On the other hand by $H \circ \mathrm{in}_1 = \lambda^n$, we have $(\lambda^n)^*\omega_{\underline{n}}(P) = \mathrm{in}_1^* H^* \omega_{\underline{n}}(P) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(1,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ for any $\boldsymbol{x} \in \Box^n$.

Second, by definition, we have $dH^*\omega_{\underline{n}}(P) = \sum_i \sum_{i_2,\cdots,i_p} \dfrac{\partial\, a_{i_2,\cdots,i_p}}{\partial\, x_i}(t,\boldsymbol{x})\, dx_i \wedge dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} \dfrac{\partial\, b_{i_1,\cdots,i_p}}{\partial\, t}(t,\boldsymbol{x})\, dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_i \sum_{i_1,\cdots,i_p} \dfrac{\partial\, b_{i_1,\cdots,i_p}}{\partial\, x_i}(t,\boldsymbol{x})\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, and hence we obtain $D_{\mathcal{U}}(d\omega)_{\underline{n}}(P) = \int_I H^*\, d\omega_{\underline{n}}(P) = -\sum_i \sum_{i_2,\cdots,i_p} \int_I \dfrac{\partial\, a_{i_2,\cdots,i_p}}{\partial\, x_i}(t,\boldsymbol{x})\, dt \cdot dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} \int_I \dfrac{\partial\, b_{i_1,\cdots,i_p}}{\partial\, t}(t,\boldsymbol{x})\, dt \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I \times \Box^n$.

Third, we have $D_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_{i_2,\cdots,i_p} \int_I a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\, dt \cdot dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, and hence we obtain $dD_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2,\cdots,i_p} \int_I \dfrac{\partial\, a_{i_2,\cdots,i_p}}{\partial\, x_i}(t,\boldsymbol{x})\, dt \cdot dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I \times \Box^n$.

Hence $[dD_{\mathcal{U}}(\omega)_{\underline{n}}(P) + D_{\mathcal{U}}(d\omega)_{\underline{n}}(P)](\boldsymbol{x}) = \sum_{i_1,\cdots,i_p} \int_I \dfrac{\partial\, b_{i_1,\cdots,i_p}}{\partial\, t}(t,\boldsymbol{x})\, dt \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(1,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p} - \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(0,\boldsymbol{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $\boldsymbol{x} \in \Box^n$. Thus we obtain $dD_{\mathcal{U}}(\omega)(P) + D_{\mathcal{U}}(d\omega)(P) = \widetilde{\omega}(P) - \omega(P)$. By the above construction of $D_{\mathcal{U}}$, it is clear to see $D_{\mathcal{U}}(\mathcal{A}^p_{\Box_c}(\mathcal{U})) \subset \mathcal{A}^{p-1}_{\Box_c}(\mathcal{U})$, and it completes the proof of the lemma. $\Box$

Remark 8.7. *We have* $b_{i_1,\cdots,i_p}(1,\boldsymbol{x}) = b_{i_1,\cdots,i_p}(0,\lambda^n(\boldsymbol{x}))\lambda'(x_{i_1}) \cdot \cdots \cdot \lambda'(x_{i_p})$ *for* $1 \le i_1 < \ldots < i_p \le n$ *and* $\boldsymbol{x} = (x_1,\ldots,x_n) \in \Box^n$, *since* $(\lambda^n)^*\omega_{\underline{n}}(P) = in_1^* H^* \omega_{\underline{n}}(P)$.

Let $\omega \in \mathcal{A}^*_{\Box}(X)$ and $P \in \mathcal{E}^X(\Box^n)$. Then a cubical complex $K = \{\sigma\,;\, \sigma < \Box^n\}$ derives cubical subdivisions $K_r = (\mathrm{Sd}^{\mathcal{U}}_P)^r(K)$ and $K_* = (\mathrm{Sd}^{\mathcal{U}}_P)^*(K)$ where $K_* = K_r$ for sufficiently large $r \ge 0$. We define $\omega^{(r)} \in \mathcal{A}^p_{\Box}(\mathcal{U})$, $r \ge 0$, as follows: for any $\sigma \in K_r$,

$$\omega^{(r)}_{\underline{n}}(P)|_{\mathrm{Int}\,\sigma} = \hat{\omega}^{(r)}_\sigma(P|_\sigma)|_{\mathrm{Int}\,\sigma},$$

where $\hat{\omega}^{(r)}_\sigma(P|_\sigma)|_{\mathrm{Int}\,\sigma} = \omega_{\underline{n}}(P|_\sigma \circ \phi_\sigma) \circ \lambda^n \circ \phi_\sigma^{-1} : \mathrm{Int}\,\sigma \overset{\phi_\sigma^{-1}}{\approx} \mathrm{Int}\,\Box^n \xrightarrow{\lambda^n} \mathrm{Int}\,\Box^n \xrightarrow{\omega_{\underline{n}}(P\circ\phi_\sigma)} \wedge^p$. Then by definition, $\omega^{(r)}_{\underline{n}}(P)|_{\mathrm{Int}\,\sigma}$ can be smoothly extended to $\partial\sigma$, and hence $\omega^{(r)}_{\underline{n}}(P) : \Box^n \to \wedge^p_{T^*_n}$ is well-defined and we obtain $\omega^{(r)} \in \mathcal{A}^p_{\Box}(X)$.

Lemma 8.8. *There is a homomorphism* $D^{(r)}_{\mathcal{U}} : \mathcal{A}^*_{\Box}(X) \to \mathcal{A}^*_{\Box}(X)$ *such that* $dD^{(r)}_{\mathcal{U}}(\omega) + D^{(r)}_{\mathcal{U}}(d\omega) = \omega^{(r+1)} - \omega^{(r)}$ *and* $D^{(r)}_{\mathcal{U}}(\mathcal{A}^p_{\Box_c}(\mathcal{U})) \subset \mathcal{A}^{p-1}_{\Box_c}(\mathcal{U})$ *for* $p \ge 0$.

Proof: For any $\omega \in \mathcal{A}^p_{\Box}(\mathcal{U})$, we define $D^{(r)}_{\mathcal{U}}(\omega) \in \mathcal{A}^p_{\Box}(X)$ as follows: let $P \in \mathcal{E}^X(\Box^n)$. We have a cubical complex $K = \{\sigma\,;\, \sigma < \Box^n\}$ which derives cubical subdivisions $K_r = (\mathrm{Sd}^{\mathcal{U}}_P)^r(K)$ of $\Box^n$ and $\widehat{K}_r = \mathrm{Td}^{\mathcal{U}}_P(K_r)$ of $I \times \Box^n$ so that $\mathrm{in}_0^*\widehat{K}_r = K_r$ and $\mathrm{in}_1^*\widehat{K}_r = K_{r+1}$. Now we define a smooth function $\widehat{\omega}(P) : I \times \Box^n \to \wedge^p(T^*_{n+1})$ as follows: for any $\sigma \in \widehat{K}^{n+1}_r$,

$$\widehat{\omega}(P)|_{\mathrm{Int}\,\sigma} = \widehat{\omega}'_\sigma(P \circ \mathrm{pr}_2|_\sigma)|_{\mathrm{Int}\,\sigma} : I \times \Box^n \longrightarrow \wedge^p(T^*_{n+1}),$$

where $\widehat{\omega}'_\sigma(P{\circ}\mathrm{pr}_2|_\sigma)|_{\mathrm{Int}\,\sigma} = \omega_{\underline{n+1}}(P{\circ}\mathrm{pr}_2|_\sigma{\circ}\phi_\sigma){\circ}\lambda^{n+1}{\circ}\phi_\sigma^{-1} : \mathrm{Int}\,\sigma \overset{\phi_\sigma^{-1}}{\approx} \mathrm{Int}\,\square^{n+1} \overset{\lambda^{n+1}}{\approx} \mathrm{Int}\,\square^{n+1} \xrightarrow{\omega_{\underline{n+1}}(P\circ\mathrm{pr}_2\circ\phi_\sigma)} \wedge^p$. Then by definition, $\widehat{\omega}'_\sigma(P{\circ}\mathrm{pr}_2|_\sigma)|_{\mathrm{Int}\,\sigma}$ can be smoothly extended to σ and we obtain a smooth function $\widehat{\omega}(P) : I{\times}\square^n \to \wedge^p_{T^*_{n+1}}$.

First, a cubical $(p{-}1)$-form $D^{(r)}_{\mathcal{U}}(\omega) \in \mathcal{A}^{p-1}_{\underline{\square}}(X)$ is defined as follows: for any cubical differential p-form $\omega : \mathcal{E}^X_{\underline{\square}} \to \wedge^p_{\underline{\square}}$ on a plot $P \in \mathcal{E}^X_{\underline{\square}}$,

$$D^{(r)}_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \int_I \widehat{\omega}(P) : \square^n \to \wedge^{p-1}(T^*_n),$$
$$\left[\int_I \widehat{\omega}(P)\right](\boldsymbol{x}) = \sum_{i_2,\cdots,i_p} \int_0^1 a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\,dt{\cdot}\,dx_{i_2} \wedge \cdots \wedge dx_{i_p},$$

where $\widehat{\omega}(P) = \sum_{i_2,\cdots,i_p} a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\,dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(t,\boldsymbol{x})\;dx_{i_1} \wedge \cdots \wedge dx_{i_p}$: $I{\times}\square^n \to \wedge^{p-1}(T^*_{n+1})$, $(t,\boldsymbol{x}) \in I{\times}\square^n$ and $T^*_{n+1} = \mathbb{R}\,dt \oplus \bigoplus_{i=1}^{n} \mathbb{R}\,dx_i$. Then, since $\mathrm{in}_0^*\widehat{K}_r = K_r$ and $\mathrm{in}_1^*\widehat{K}_r = K_{r+1}$, we easily see that $\omega^{(r)}_{\underline{n}}(P) = \mathrm{in}_0^*\,\widehat{\omega}(P) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(0,\boldsymbol{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $\omega^{(r+1)}_{\underline{n}}(P) = \mathrm{in}_1^*\,\widehat{\omega}(P) = \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(1,\boldsymbol{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Second, by definition, we have $\widehat{d\omega}(P) = d\widehat{\omega}(P) = \sum_i \sum_{i_2,\cdots,i_p} \dfrac{\partial\,a_{i_2,\cdots,i_p}}{\partial\,x_i}(t,\boldsymbol{x})\;dx_i \wedge dt \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} \dfrac{\partial\,b_{i_1,\cdots,i_p}}{\partial\,t}(t,\boldsymbol{x})\,dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} + \sum_i \sum_{i_1,\cdots,i_p} \dfrac{\partial\,b_{i_1,\cdots,i_p}}{\partial\,x_i}(t,\boldsymbol{x})\,dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, and hence $D^{(r)}_{\mathcal{U}}(d\omega)_{\underline{n}}(P) = \int_I \widehat{d\omega}(P) = -\sum_i \sum_{i_2,\cdots,i_p} \int_I \dfrac{\partial\,a_{i_2,\cdots,i_p}}{\partial\,x_i}(t,\boldsymbol{x})\,dt \cdot dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} + \sum_{i_1,\cdots,i_p} \int_I \dfrac{\partial\,b_{i_1,\cdots,i_p}}{\partial\,t}(t,\boldsymbol{x})\,dt{\cdot}\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I{\times}\square^n$.

Third, we have $D^{(r)}_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_{i_2,\cdots,i_p} \int_I a_{i_2,\cdots,i_p}(t,\boldsymbol{x})\,dt{\cdot}\,dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, and hence we obtain $dD^{(r)}_{\mathcal{U}}(\omega)_{\underline{n}}(P) = \sum_i \sum_{i_2,\cdots,i_p} \int_I \dfrac{\partial\,a_{i_2,\cdots,i_p}}{\partial\,x_i}(t,\boldsymbol{x})\,dt{\cdot}\,dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $(t,\boldsymbol{x}) \in I{\times}\square^n$.

Hence $\left[dD^{(r)}_{\mathcal{U}}(\omega)_{\underline{n}}(P) + D^{(r)}_{\mathcal{U}}(d\omega)_{\underline{n}}(P)\right](\boldsymbol{x}) = \sum_{i_1,\cdots,i_p} \int_I \dfrac{\partial\,b_{i_1,\cdots,i_p}}{\partial\,t}(t,\boldsymbol{x})\,dt{\cdot}\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ $= \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(1,\boldsymbol{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_p} - \sum_{i_1,\cdots,i_p} b_{i_1,\cdots,i_p}(0,\boldsymbol{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $\boldsymbol{x} \in \square^n$. Thus we obtain $dD^{(r)}_{\mathcal{U}}(\omega)(P) + D^{(r)}_{\mathcal{U}}(d\omega)(P) = \omega^{(r+1)}(P) - \omega^{(r)}(P)$. By the above construction of $D^{(r)}_{\mathcal{U}}$, it is clear to see that $D^{(r)}_{\mathcal{U}}(\mathcal{A}^p_{\underline{\square}_c}(\mathcal{U})) \subset \mathcal{A}^{p-1}_{\underline{\square}_c}(\mathcal{U})$. □

Theorem 8.9. *The restriction* res : $\mathcal{A}^*_{\underline{\square}}(X) \to \mathcal{A}^*_{\underline{\square}}(\mathcal{U})$ *induces an isomorphism of cubical de Rham cohomologies* res* : $H^*_{\underline{\square}}(X) \to H^*_{\underline{\square}}(\mathcal{U})$. *In addition,* res *induces a map* res : $\mathcal{A}^*_{\underline{\square}_c}(X) \to \mathcal{A}^*_{\underline{\square}_c}(\mathcal{U})$ *which further induces an isomorphism* res* : $H^*_{\underline{\square}_c}(X) \to H^*_{\underline{\square}_c}(\mathcal{U})$.

Proof: For any $\omega \in \mathcal{A}^p_\square(\mathcal{U})$, we define $\omega^* \in \mathcal{A}^p_\square(X)$ as follows: let $P \in \mathcal{E}^X(\square^n)$. Then we obtain a cubical complex $K = \{\sigma\,;\,\sigma < \square^n\}$ which derives a cubical subdivision $K_* = (\mathrm{Sd}^{\mathcal{U}}_P)^*(K)$. We define cubical differential p-forms $\omega^* \in \mathcal{A}^p_\square(\mathcal{U})$ as follows: for any $\sigma \in K_*$,

$$\omega^*_{\underline{n}}(P)|_{\mathrm{Int}\,\sigma} = \hat{\omega}^*_\sigma(P|_\sigma)|_{\mathrm{Int}\,\sigma},$$

where $\hat{\omega}^*_\sigma(P|_\sigma)|_{\mathrm{Int}\,\sigma} = \omega_{\underline{n}}(P|_\sigma\circ\phi_\sigma)\circ\lambda^n\circ\phi_\sigma^{-1} : \mathrm{Int}\,\sigma \overset{\phi_\sigma^{-1}}{\approx} \mathrm{Int}\,\square^n \overset{\lambda^n}{\approx} \mathrm{Int}\,\square^n \xrightarrow{\omega_{\underline{n}}(P\circ\phi_\sigma)} \wedge^p$. Then by definition, $\omega^*_{\underline{n}}(P)|_{\mathrm{Int}\,\sigma}$ can be uniquely extended to $\partial\sigma$ and we obtain $\omega^*_{\underline{n}}(P) : \square^n \to \wedge^p_{T^*_n}$ so that $\omega^* \in \mathcal{A}^p_\square(X)$ whose restriction to $\mathcal{A}^p_\square(\mathcal{U})$ equals, by definition, to $\widetilde{\omega}$ with a $(p-1)$-form $D_{\mathcal{U}}(\omega) \in \mathcal{A}^{p-1}_\square(\mathcal{U})$ satisfying $d\,D_{\mathcal{U}}(\omega) = \widetilde{\omega} - \omega$ if $d\omega = 0$, by Lemma 8.6. If $d\omega = 0$, then $d\hat{\omega}^* = 0$, and hence $d\omega^* = 0$. Thus the restriction res : $\mathcal{A}^*_\square(X) \to \mathcal{A}^*_\square(\mathcal{U})$ induces an epimorphism $\mathrm{res}^* : H^*_\square(X) \to H^*_\square(\mathcal{U})$ of cubical de Rham cohomologies.

So we are left to show that $\mathrm{res}^* : H^*_\square(X) \to H^*_\square(\mathcal{U})$ is a monomorphism: let $\omega \in \mathcal{A}^p_\square(X)$. Then we obtain a cubical differential p-forms $\omega^{(r)} \in \mathcal{A}^p_\square(\mathcal{U})$ and $\omega^* \in \mathcal{A}^p_\square(\mathcal{U})$ so that $\omega^{(r)} = \omega^*$ for sufficiently large $r \geq 0$. By Lemma 8.8, there is a $(p-1)$-form $D^{(r)}_{\mathcal{U}}(\omega) \in \mathcal{A}^{p-1}_\square(X)$ such that $d\,D^{(r)}_{\mathcal{U}}(\omega) = \omega^{(r+1)} - \omega^{(r)}$ if $d\omega = 0$. If we assume $\mathrm{res}^*([\omega]) = 0$, then we may assume $\mathrm{res}(\omega) = 0$ and $d\omega = 0$, and so we obtain $\omega^* = 0$ and $\omega = d\left\{\sum_{r=0}^{N} D^{(r)}_{\mathcal{U}}(\omega) - D_{\{X\}}(\omega)\right\}$ for sufficiently large $N \geq 0$, in other words, ω is an exact form and cohomologous to zero. Thus $\mathrm{res}^* : H^*_\square(X) \to H^*_\square(\mathcal{U})$ is an monomorphism. □

9. Mayer-Vietoris sequence and Theorem of de Rham

Theorem 9.1. *Let $\mathcal{U} = \{U_1, U_2\}$ be any open covering of a differentiable space X. The canonical inclusions $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce $\psi^\natural : \mathcal{A}^p_\square(\mathcal{U}) \to \mathcal{A}^p_\square(U_1) \oplus \mathcal{A}^p_\square(U_2)$ and $\phi^\natural : \mathcal{A}^p_\square(U_1) \oplus \mathcal{A}^p_\square(U_2) \to \mathcal{A}^p_\square(U_1 \cap U_2)$ by $\psi^\natural(\omega) = i_1^\sharp\omega \oplus i_2^\sharp\omega$ and $\phi^\natural(\eta_1 \oplus \eta_2) = j_1^\sharp\eta_1 - j_2^\sharp\eta_2$. Then we obtain the following long exact sequence.*

$$H^0_\square(X) \to \cdots \to H^p_\square(X) \xrightarrow{\psi^*} H^p_\square(U_1) \oplus H^p_\square(U_2) \xrightarrow{\phi^*} H^p_\square(U_1 \cap U_2)$$
$$\to H^{p+1}_\square(X) \xrightarrow{\psi^*} H^{p+1}_\square(U_1) \oplus H^{p+1}_\square(U_2) \xrightarrow{\phi^*} H^{p+1}_\square(U_1 \cap U_2) \to \cdots,$$

where ψ^* and ϕ^* are induced from $\psi^\natural$ and $\phi^\natural$.

Proof: Since $H^*_\square(X) = H^*_\square(\mathcal{U})$ by Theorem 8.9, we are left to show long exact sequence

$$0 \longrightarrow \mathcal{A}^p_\square(\mathcal{U}) \xrightarrow{\psi^\natural} \mathcal{A}^p_\square(U_1) \oplus \mathcal{A}^p_\square(U_2) \xrightarrow{\phi^\natural} \mathcal{A}^p_\square(U_0) \longrightarrow 0, \quad U_0 = U_1 \cap U_2.$$

(exactness at $\mathcal{A}^p_\square(\mathcal{U})$): Assume $\psi^\natural(\omega) = 0$, and so $j_t^\sharp\omega = 0$ for $t = 1, 2$. Then for any $P : \square^n \to X$, $P \in \mathcal{E}^{\mathcal{U}}_\square$, we have either $\mathrm{Im}\,P \subset U_1$ or $\mathrm{Im}\,P \subset U_2$. Therefore, we may assume either $P \in \mathcal{E}^{U_0}_\square$ or $P \in \mathcal{E}^{U_1}_\square$. In each case, we have $\omega_{\underline{n}}(P) = 0$, which implies that $\omega = 0$. Thus $\psi^\natural$ is monic.

(exactness at $\mathcal{A}^p_{\square}(U_1) \oplus \mathcal{A}^p_{\square}(U_2)$): Assume $\phi^\natural(\eta^{(1)} \oplus \eta^{(2)}) = 0$, and so $i_1^\sharp \eta^{(1)} = i_2^\sharp \eta^{(2)}$. Then we can construct a differential p-form $\omega \in \mathcal{A}^p_{\square}(\mathcal{U})$ as follows: for any $P \in \mathcal{E}^{\mathcal{U}}_{\square}$, we have $\operatorname{Im} P \subset U_t$ for either $t = 1$ or 2. Using this t, we define $\omega_{\underline{n}}(P) = \eta^{(t)}_{\underline{n}}(P)$. If $\operatorname{Im} P \subset U_1$ and $\operatorname{Im} P \subset U_2$, then we have $\operatorname{Im} P \subset U_1 \cap U_2$, and hence $\eta^{(1)}_{\underline{n}}(P) = \eta^{(2)}_{\underline{n}}(P)$, since $i_1^\sharp \eta^{(1)} = i_2^\sharp \eta^{(2)}$. It implies that ω is well-defined and that $\psi^\natural(\omega) = \eta^{(1)} \oplus \eta^{(2)}$. The converse is clear and we have $\operatorname{Ker} \phi^\natural = \operatorname{Im} \psi^\natural$.

(exactness at $\mathcal{A}^p_{\square}(U_0)$): Assume $\kappa \in \mathcal{A}^p_{\square}(U_0)$. We define $\kappa^{(t)} \in \mathcal{A}^p_{\square}(U_t)$, $t = 1, 2$ as follows: for any $P_t \in \mathcal{E}^{U_t}_{\square}$, we define $\kappa^{(t)}_{\underline{n}}(P_t)(\boldsymbol{x})$ by $(-1)^{t-1} \rho^{(3-t)}_{P_t}(\boldsymbol{x}) \cdot \kappa_{\underline{n}}(P_t)(\boldsymbol{x})$ if $\boldsymbol{x} \in P_t^{-1}(U_{3-t})$ and by 0 if $\boldsymbol{x} \notin \operatorname{Supp} \rho^{3-t}_{P_t}$. Hence $\kappa^{(t)}$ is well-defined satisfying $i_1^\sharp \kappa^{(1)} - i_2^\sharp \kappa^{(2)} = \kappa$, and we obtain $\kappa = \phi^\natural(\kappa^{(1)} \oplus \kappa^{(2)})$. Thus $\phi^\natural$ is an epimorphism.

Since $\psi^\natural$ and $\phi^\natural$ are clearly cochain maps, we obtain the desired long exact sequence. $\square$

Now let us turn our attention to the differential forms with compact support. Let $X = (X, \mathcal{E}^X)$ be a weakly-separated differentiable space.

Definition 9.2. *Let U be an open set in X, $F \subset U$ a closed set in X and $\mathcal{U}$ an open covering of U. We denote by $\mathcal{A}^p_{\square_c}(\mathcal{U}; F)$ the set of all $\omega \in \mathcal{A}^p_{\square_c}(\mathcal{U})$ satisfying $\operatorname{Supp} \omega_{\underline{n}}(P) \subset P^{-1}(F)$ for any $P \in \mathcal{E}(\square^n)$. For example, any $\omega \in \mathcal{A}^p_{\square_c}(\mathcal{U})$ is in $\mathcal{A}^p_{\square_c}(\mathcal{U}; F)$ if $F \supset K_\omega$. We denote by $H^*_{\square_c}(\mathcal{U}; F)$ the cohomology of $\mathcal{A}^*_{\square_c}(\mathcal{U}; F)$ a differential subalgebra of $\mathcal{A}^*_{\square_c}(\mathcal{U})$.*

Definition 9.3. *Let U and V be open sets and $F \subset U$ and $G \subset V$ be closed sets in X so that $(U, F) \subset (V, G)$, and $j : (U, F) \hookrightarrow (V, G)$ be the canonical inclusion. Let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of U and V, respectively, satisfying $F \cap W = \emptyset$ for any $W \in \mathcal{V} \smallsetminus \mathcal{U}$. Then a homomorphism $j_\sharp : \mathcal{A}^p_{\square_c}(\mathcal{U}; F) \to \mathcal{A}^p_{\square_c}(\mathcal{V}; G)$ is defined as follows: for any $\omega \in \mathcal{A}^p_{\square_c}(\mathcal{U}; F)$, $j_\sharp \omega \in \mathcal{A}^p_{\square_c}(\mathcal{V}; G)$ is given, for $Q \in \mathcal{E}^{\mathcal{V}}(\square^m)$, by*

$$\begin{cases} (j_\sharp \omega)_{\underline{m}}(Q) = \omega_{\underline{m}}(Q), & \text{if } \operatorname{Im} Q \subset W \text{ for some } W \in \mathcal{U}, \\ (j_\sharp \omega)_{\underline{m}}(Q) = 0, & \text{if } \operatorname{Im} Q \subset W \text{ for some } W \in \mathcal{V} \smallsetminus \mathcal{U} \end{cases}$$

with $K_{j_\sharp \omega} = K_\omega \subset F \subset G$. In particular, for any $\omega \in \mathcal{A}^p_{\square_c}(\mathcal{U})$, we have $\omega \in \mathcal{A}^p_{\square_c}(\mathcal{U}; K_\omega)$, and so we obtain $j_\sharp \omega \in \mathcal{A}^p_{\square_c}(j_\sharp \mathcal{U}_\omega; K_\omega) \subset \mathcal{A}^p_{\square_c}(j_\sharp \mathcal{U}_\omega)$, $j_\sharp \mathcal{U}_\omega = \mathcal{U} \cup \{V \smallsetminus K_\omega\}$.

Remark 9.4. *In Definition 9.3, the map $j_\sharp$ induced from $j : (U, F) \hookrightarrow (V, G)$ satisfies that $(j_\sharp \omega)_{\underline{m}}(j \circ Q) = \omega_{\underline{m}}(Q)$ for any $m \geq 0$ and $Q \in \mathcal{E}^{\mathcal{U}}(\square^m)$.*

Proposition 9.5. *Let $X = (X, \mathcal{E}^X)$ be a weakly-separated differentiable space and U and V open in X. Then the correspondence $\mathcal{A}^*_{\square_c}(U) \ni \omega \mapsto j_\sharp \omega \in \mathcal{A}^*_{\square_c}(j_\sharp \mathcal{U}_\omega)$ induced from the canonical inclusion $j : U \hookrightarrow V$ induces a homomorphism $j_* : H^*_{\square_c}(U) \to H^*_{\square_c}(V)$, since there is a canonical isomorphism $H^*_{\square_c}(j_\sharp \mathcal{U}_\omega) \cong H^*_{\square_c}(V)$ by Theorem 8.9.*

Proof: Let $\omega, \eta \in \mathcal{A}^*_{\square_c}(U)$. Then $K = K_\omega \cup K_\eta$ is compact in U and hence in X. Let $\mathcal{U} = \{U, V \smallsetminus K\}$, which is a finer open covering of $\mathcal{U}_\omega$ and $\mathcal{U}_\eta$, and hence both isomorphisms $H^*_{\square_c}(V) \to H^*_{\square_c}(\mathcal{U}_\omega)$ and $H^*_{\square_c}(V) \to H^*_{\square_c}(\mathcal{U}_\eta)$ defined in Theorem 8.9 go through the isomorphism $H^*_{\square_c}(V) \to H^*_{\square_c}(\mathcal{U})$. Thus the homomorphisms $H^*_{\square_c}(\mathcal{U}) \to H^*_{\square_c}(\mathcal{U}_\omega)$ and $H^*_{\square_c}(\mathcal{U}) \to H^*_{\square_c}(\mathcal{U}_\eta)$ are also isomorphisms. By definition, $j_\sharp(\omega+\eta) = j_\sharp(\omega) + j_\sharp(\eta)$ in $\mathcal{A}^*_{\square_c}(\mathcal{U})$, and hence $j_*([\omega+\eta]) = j_*([\omega]) + j_*([\eta])$ in $H^*_{\square_c}(X)$ for any $[\omega], [\eta] \in H^*_{\square_c}(U)$. □

Theorem 9.6. *Let $\mathcal{U} = \{U_1, U_2\}$ be an open covering of a weakly-separated differentiable space X with a normal partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$ belonging to $\mathcal{U}$, i.e., there are closed subsets $\{G_1, G_2\}$ such that $G_t \subset U_t$ and $\operatorname{Supp} \rho^{(t)}_{\underline{n}}(P) \subset P^{-1}(G_t)$ for any $P \in \mathcal{E}(\square^n)$, $t = 1, 2$. Then we have $G_1 \cup G_2 = X$. Let $G_0 = G_1 \cap G_2 \subset U_0 = U_1 \cap U_2$. The canonical inclusions $i_t : U_1 \cap U_2 \hookrightarrow U_t$ and $j_t : U_t \hookrightarrow X$, $t = 1, 2$, induce $\phi_* : H^p_{\square_c}(U_0) \to H^p_{\square_c}(U_1) \oplus H^p_{\square_c}(U_2)$ and $\psi_* : H^p_{\square_c}(U_1) \oplus H^p_{\square_c}(U_2) \to H^p_{\square_c}(X)$ by $\phi_*([\omega]) = i_{1*}[\omega] \oplus i_{2*}[\omega]$ and $\psi_*([\eta_1] \oplus [\eta_2]) = j_{1*}[\eta_1] - j_{2*}[\eta_2]$. Then we obtain the following long exact sequence.*

$$H^0_{\square_c}(U_0) \to \cdots \to H^p_{\square_c}(U_0) \xrightarrow{\phi_*} H^p_{\square_c}(U_1) \oplus H^p_{\square_c}(U_2) \xrightarrow{\psi_*} H^p_{\square_c}(X)$$
$$\xrightarrow{d_*} H^{p+1}_{\square_c}(U_0) \xrightarrow{\phi_*} H^{p+1}_{\square_c}(U_1) \oplus H^{p+1}_{\square_c}(U_2) \xrightarrow{\psi_*} H^{p+1}_{\square_c}(X) \to \cdots .$$

Proof: For any closed subsets $G'_t \supset G_t$ in U_t, there is a following short exact sequence.

$$0 \longrightarrow \mathcal{A}^p_{\square_c}(\mathcal{U}_0; G'_0) \xrightarrow{\phi_\natural} \mathcal{A}^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus \mathcal{A}^p_{\square_c}(\mathcal{U}_2; G'_2) \xrightarrow{\psi_\natural} \mathcal{A}^p_{\square_c}(\mathcal{U}_3; X) \longrightarrow 0,$$

where $G'_0 = G'_1 \cap G'_2$, $\mathcal{U}_0 = \{U_0\}$, $\mathcal{U}_t = \{U_0, U_t \smallsetminus G'_{3-t}\}$, $t = 1, 2$ and $\mathcal{U}_3 = \{U_0, U_1 \smallsetminus G'_2, U_2 \smallsetminus G'_1\}$, which are open coverings of U_0, U_t and X, respectively.

- **(exactness at $\mathcal{A}^p_{\square_c}(\mathcal{U}_0; G'_0)$):** Assume $\phi_\natural(\omega) = 0$. Then $i_{1\sharp}(\omega) = i_{2\sharp}(\omega) = 0$. Since $i_{1\sharp}(\omega)$ is an extension of ω, we obtain $\omega = 0$. Thus $\phi_\natural$ is a monomorphism.
- **(exactness at $\mathcal{A}^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus \mathcal{A}^p_{\square_c}(\mathcal{U}_2; G'_2)$):** Assume $\psi_\natural(\eta^{(1)} \oplus \eta^{(2)}) = 0$. Then we have $j_{1\sharp}(\eta^{(1)}) = j_{2\sharp}(\eta^{(2)})$. For any plot $P : \square^n \to X$, we obtain $j_{1\sharp}(\eta^{(1)})_{\underline{n}}(P) = j_{2\sharp}(\eta^{(2)})_{\underline{n}}(P)$. So, for any plot $Q : \square^m \to U_0$, $\eta^{(1)}_B(i_1 \circ Q) = j^\sharp_1 \eta^{(1)}_{\underline{m}}(j_1 \circ i_1 \circ Q) = j^\sharp_2 \eta^{(2)}_{\underline{m}}(j_2 \circ i_2 \circ Q) = \eta^{(2)}_{\underline{m}}(i_2 \circ Q)$. Then, we define $\eta^{(0)} \in \mathcal{A}^p_{\square}(U_0)$ by $\eta^{(0)}_{\underline{m}}(Q) = \eta^{(1)}_{\underline{m}}(i_1 \circ Q) = \eta^{(2)}_{\underline{m}}(i_2 \circ Q)$. On the other hand, $K_{j_{t\sharp}\eta^{(t)}} = K_{\eta^{(t)}}$ by definition, and hence we obtain
$$\operatorname{Supp} \eta^{(0)}_{\underline{m}}(Q) = \operatorname{Supp} \eta^{(1)}_{\underline{m}}(i_1 \circ Q) = \operatorname{Supp} \eta^{(2)}_{\underline{m}}(i_2 \circ Q) \subset Q^{-1}(K_{\eta^{(1)}} \cap K_{\eta^{(2)}}).$$
Then we have $\eta^{(0)} \in \mathcal{A}^p_{\square_c}(\mathcal{U}_0)$, for $K_{\eta^{(0)}} = K_{\eta^{(1)}} \cap K_{\eta^{(2)}}$ is compact in U_0, which satisfies $\phi_\natural(\eta^{(0)}) = (\eta^{(1)}, \eta^{(2)})$. Thus $(\eta^{(1)}, \eta^{(2)})$ is in the image of $\phi_\natural$. The other direction is clear by definition and it implies the exactness at $\mathcal{A}^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus \mathcal{A}^p_{\square_c}(\mathcal{U}_2; G'_2)$.
- **(exactness at $\mathcal{A}^p_{\square_c}(\mathcal{U}_3; X)$):** Assume $\kappa \in \mathcal{A}^p_{\square_c}(\mathcal{U}_3; X)$. For any plot $P_t : \square^{n_t} \to U_t$, we define $\kappa^{(t)}_{\underline{n_t}}(P_t)(\boldsymbol{x})$ by $(-1)^{t-1} \rho^{(t)}_{\underline{n_t}}(P_t)(\boldsymbol{x}) \cdot \kappa_{\underline{n_t}}(j_t \circ P_t)(\boldsymbol{x})$ if $\boldsymbol{x} \in P_t^{-1}(U_0)$ and by 0

if $\boldsymbol{x} \notin \operatorname{Supp} \rho^{(t)}_{\underline{n_t}}(P_t)$. Then $\kappa^{(t)}$ is a differential p-form on U_t and $\kappa^{(t)} \in \mathcal{A}^p_{\square_c}(U_t)$ for $K_{\kappa^{(t)}} = K_\kappa \cap G_t \subset G'_t$ is compact in U_t. Then we have $\psi_\natural(\kappa^{(1)} \oplus \kappa^{(2)}) = \kappa$, and hence κ is in the image of $\psi_\natural$. Thus $\psi_\natural$ is an epimorphism.

Since $\phi_\natural$ and $\psi_\natural$ are clearly cochain maps, we obtain the following long exact sequence.

$$\begin{aligned} H^0_{\square_c}(\mathcal{U}_0; G'_0) \to \cdots \to H^p_{\square_c}(\mathcal{U}_0; G'_0) &\xrightarrow{\overline{\phi}_*} H^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus H^p_{\square_c}(\mathcal{U}_2; G'_2) \xrightarrow{\overline{\psi}_*} H^p_{\square_{\square_c}}(\mathcal{U}_3) \\ &\xrightarrow{\overline{d}_*} H^{p+1}_{\square_c}(\mathcal{U}_0; G'_0) \xrightarrow{\overline{\phi}_*} H^{p+1}_{\square_c}(\mathcal{U}_1; G'_1) \oplus H^{p+1}_{\square_c}(\mathcal{U}_2; G'_2) \xrightarrow{\overline{\psi}_*} H^{p+1}_{\square_{\square_c}}(\mathcal{U}_3) \to \cdots. \end{aligned}$$

So we can define connecting homomorphism $d_* : H^p_{\square_c}(X) \overset{\mathrm{res}^*}{\cong} H^p_{\square_c}(\mathcal{U}_3) \xrightarrow{\overline{d}_*} H^{p+1}_c(\mathcal{U}_0; G'_0) \to H^{p+1}_{\square_c}(U_0)$ where the latter map is induced from the natural inclusion $\mathcal{A}^{p+1}_{\square_c}(\mathcal{U}_0; G'_0) \subset \mathcal{A}^{p+1}_{\square_c}(\mathcal{U}_0) = \mathcal{A}^{p+1}_{\square_c}(U_0)$, which fits in with the following commutative ladder.

$$\begin{array}{ccccccc} H^p_{\square_c}(\mathcal{U}_0; G'_0) & \xrightarrow{\overline{\phi}_*} & H^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus H^p_{\square_c}(\mathcal{U}_2; G'_2) & \xrightarrow{\overline{\psi}_*} & H^p_{\square_c}(\mathcal{U}_3) & \xrightarrow{\overline{d}_*} & H^{p+1}_{\square_c}(\mathcal{U}_0; G'_0) \\ \downarrow & & \downarrow & & \| & & \downarrow \\ H^p_{\square_c}(\mathcal{U}_0) & & H^p_{\square_c}(\mathcal{U}_1) \oplus H^p_{\square_c}(\mathcal{U}_2) & & H^p_{\square_c}(\mathcal{U}_3) & & H^{p+1}_{\square_c}(\mathcal{U}_0) \\ \| & & \cong \uparrow \mathrm{res}^* \oplus \mathrm{res}^* & & \cong \uparrow \mathrm{res}^* & & \| \\ H^p_{\square_c}(U_0) & \xrightarrow{\phi_*} & H^p_{\square_c}(U_1) \oplus H^p_{\square_c}(U_2) & \xrightarrow{\psi_*} & H^p_{\square_c}(X) & \xrightarrow{d_*} & H^{p+1}_{\square_c}(U_0) \end{array}$$

Using these diagrams, we show the desired exactness as follows.

(exactness at $H^p_{\square_c}(U_0)$): Assume $\phi_*([\omega]) = 0$. Let $G'_t = G_t \cup K_\omega$, $t = 0, 1, 2$. Then $[\omega] \in H^p_{\square_c}(\mathcal{U}_0; G'_0)$ satisfying $\overline{\phi}_*([\omega])$ is zero in $H^p_{\square_c}(\mathcal{U}_1) \oplus H^p_{\square_c}(\mathcal{U}_2)$. Hence there is $\sigma^{(1)} \oplus \sigma^{(2)} \in \mathcal{A}^p_{\square_c}(\mathcal{U}_1) \oplus \mathcal{A}^p_{\square_c}(\mathcal{U}_2)$ such that $d\sigma^{(1)} \oplus d\sigma^{(2)} = \phi_\natural(\omega)$. Then we may expand G'_t as $G'_t = G_t \cup K_\omega \cup K_{\sigma^{(t)}}$, $t = 1, 2$ and $G'_0 = G'_1 \cap G'_2$, so that we obtain $\overline{\phi}_*([\omega]) = 0$, and hence $[\omega] \in \operatorname{Im} \overline{d}_*$ in $H^{p+1}_{\square_c}(\mathcal{U}_0; G'_0)$. Thus $[\omega]$ is in the image of d_*.

(exactness at $H^p_{\square_c}(U_1) \oplus H^p_{\square_c}(U_2)$): Assume $\psi_*([\eta^{(1)}] \oplus [\eta^{(2)}]) = 0$. Let $G'_t = G_t \cup K_{\eta^{(t)}}$, $t = 1, 2$ and $G'_0 = G'_1 \cap G'_2$, so that $[\eta^{(1)}] \oplus [\eta^{(2)}] \in H^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus H^p_{\square_c}(\mathcal{U}_2; G'_2)$ and $\overline{\psi}_*([\eta^{(1)}] \oplus [\eta^{(2)}]) = 0$ in $H^p_{\square_c}(\mathcal{U}_3) \cong H^p_{\square_c}(X)$. Then we obtain $[\eta^{(1)}] \oplus [\eta^{(2)}] \in \operatorname{Im} \overline{\phi}_*$ in $H^p_{\square_c}(\mathcal{U}_1; G'_1) \oplus H^p_{\square_c}(\mathcal{U}_2; G'_2)$, and hence $[\eta^{(1)}] \oplus [\eta^{(2)}]$ is in the image of ϕ_*.

(exactness at $H^p_{\square_c}(X)$): Assume $d_*([\kappa]) = 0$. Then there is $\sigma \in \mathcal{A}^p_{\square_c}(U_0)$ such that $d_\natural(\kappa) = d\sigma$ in $\mathcal{A}^{p+1}_{\square_c}(U_0)$. Let $G'_t = G_t \cup K_\sigma$, $t = 0, 1, 2$. Then we may assume $\sigma \in \mathcal{A}^p_{\square_c}(\mathcal{U}_0; G'_0)$ satisfying $d_\natural(\kappa) = d\sigma$ in $\mathcal{A}^{p+1}_{\square_c}(\mathcal{U}_0; G'_0)$, and hence $[\kappa] \in \operatorname{Im} \overline{\psi}_*$ in $H^p_{\square_c}(\mathcal{U}_3)$. Thus $[\kappa]$ is in the image of ψ_*.

The other directions are clear by definition, and it completes the proof of the theorem. $\square$

Let $\mathsf{Topology}$ be the category of topological spaces and continuous maps. Then there are natural embeddings $\mathsf{Topology} \hookrightarrow \mathsf{Differentiable}$ and $\mathsf{Topology} \hookrightarrow \mathsf{Diffeology}$.

Let $X = (X, \{X^{(n)}\,;\, n \geq -1\})$ be a topological CW complex embedded in the category $\mathsf{Diffeology}$ or $\mathsf{Differentiable}$ with the set of n-balls $\{B^n_j\}$ indexed by $j \in J_n$. Then we have open sets $U = X^{(n)} \smallsetminus X^{(n-1)}$ and $V = X^{(n)} \smallsetminus (\bigcup_{j \in J_n} \{\mathbf{0}_j\})$ in $X^{(n)}$, where $\mathbf{0}_j \in B^n_j$ denotes the element corresponding to $\mathbf{0} \in B^n = \{\boldsymbol{x} \in \mathbb{R}^n\,;\, \|\boldsymbol{x}\| \leq 1\}$ the origin of $\mathbb{R}^n$.

A ball $B^n_j = B^n$ (if we disregard the indexing) has a nice open covering given by $\{\mathrm{Int}\, B^n_j, B^n_j \smallsetminus \{\mathbf{0}\}\}$ with a partition of unity $\{\rho^{(j)}_1, \rho^{(j)}_2\}$ as follows: $\rho^{(j)}_1 = 1 - \rho^{(j)}_2$ and $\rho^{(j)}_2(\boldsymbol{x}) = \lambda(\|\boldsymbol{x}\|)$ for small $a > 0$. Thus $\mathcal{U} = \{U, V\}$ is a nice open covering of $X^{(n)}$ with a normal partition of unity $\{\rho^U, \rho^V\}$ in which ρ^U is a zero-extension of $\rho^{(j)}_1$'s on the union of balls and $\rho^V = 1 - \rho^U$. Then U is smoothly homotopy equivalent to discrete points each of which is $\mathbf{0}_j \in B^n_j$ for some $j \in J_n$ and V is smoothly homotopy equivalent to $X^{(n-1)}$. By comparing Mayer-Vietoris sequences associated to $\mathcal{U}$ in Theorem 2.3 with that in Theorem 9.1 for $X = X^{(n)}$, we obtain the following result using Remark 1.18 together with so-called five lemma, by using standard homological methods inductively on n.

Theorem 9.7. *For a CW complex X, there are natural isomorphisms*

$$H^q_{\mathcal{D}}(X) \cong H^q_{\mathcal{C}}(X) \cong H^q_{\square}(X) \cong H^q(X, \mathbb{R}) \cong \mathrm{Hom}(H_q(X), \mathbb{R}),$$

for any $q \geq 0$, and hence we have $H^1_{\mathcal{D}}(X) \cong H^1_{\mathcal{C}}(X) \cong H^1_{\square}(X) \overset{\rho}{\cong} \mathrm{Hom}(\pi_1(X), \mathbb{R})$.

Conjecture 9.8. *For a CW complex X, there are natural isomorphisms*

$$H^q_{\mathcal{D}_c}(X) \cong H^q_{\mathcal{C}_c}(X) \cong H^q_{\square_c}(X), \quad \text{for any } q \geq 0.$$

It would be possible to determine $H^*_{\square}(X)$ and $H^*_{\square_c}(X)$ by using standard methods in algebraic topology even if X is not a topological CW complex, while we do not know how to determine $H^*_{\mathcal{D}}(X)$, $H^*_{\mathcal{C}}(X)$, $H^*_{\mathcal{D}_c}(X)$ nor $H^*_{\mathcal{C}_c}(X)$, if we do not find out any appropriate nice open covering (with a normal partition of unity) on X.

10. Application to the loop space of a finite CW complex

Let X be a CW complex. Then by Theorem 9.7, de Rham cohomology $H^*_{DR}(X) = H^*_{C}(X)$ is isomorphic with the rational cohomology $H^*(X; \mathbb{R})$. Let us assume further that X is a 1-connected finite CW complex whose cell structure gives its homology decomposition. Then by Toda [12, 13], we may assume that X is a standard CW complex equipped with a infinite-dimensional CW complex $\omega(X)$ such that the inclusion $\omega(X) \hookrightarrow \Omega(X)$ is a homotopy equivalence. Thus we also have an isomorphism $H^*_{DR}(\omega(X)) \cong H^*(\Omega(X); \mathbb{R})$.

On the other hand, following Chen's arguments, we can observe de Rham complex as follows: there is a homology connection (ω, δ) on $\mathcal{A}^*_{DR}(X)$ together with a transport

T. Then we have a holonomy homomorphism $\Theta : C_*(\Omega(X))\otimes\mathbb{R} \to \mathbb{R}\langle\langle X_1, ..., X_m\rangle\rangle$ the completion by augmentation ideal of tensor algebra on X_i's which are corresponding to the module generators of $\bar{H}^*(X;\mathbb{R}) \cong \bar{H}^*_{DR}(X)$. Then we can see that Θ induces an isomorphism of de Rham cohomology and the rational cohomology of $\Omega(X)$.

Appendix A. Smooth CW complex

A smooth CW complex $X = (X, \{X^{(n)}\}_{n\geq -1})$ is a differentiable or diffeological space built up from $X^{(-1)} = \emptyset$ by inductively attaching n-balls $\{B^n_j\}_{j\in J_n}$ by C^∞ maps from their boundary spheres $\{S^{n-1}_j\}_{j\in J_n}$ to $n-1$-skeleton $X^{(n-1)}$ to obtain n-skeleton $X^{(n)}$, $n \geq 0$, where the smooth structures of balls and spheres are given by their manifold structures. Thus a plot in $X^{(n)}$ is a map $P : A \to X$ with an open covering $\{A_\alpha\}_{\alpha\in\Lambda}$ of A such that, for any α, $P(A_\alpha)$ is in $X^{(n-1)}$ or B^n_j for some $j \in J_n$ and $P|_{A_\alpha}$ is a plot of $X^{(n-1)}$ or B^n_j, respectively. Then as the colimit of $\{X^{(n)}\}$, X exists in Differentiable or Diffeology.

For a given CW complex, we can deform attaching maps of n-balls from their boundary spheres $\{S^{n-1}_j\}_{j\in J_n}$ to $n-1$-skeleton $X^{(n-1)}$ to be C^∞ maps, and obtain the following.

Theorem A.1. *A CW complex is homotopy equivalent to a smooth CW complex as topological spaces. Thus we may assume that any CW complex is smooth up to homotopy.*

Let $X = (X, \{X^{(n)}\})$ be a smooth CW complex in either Differentiable or Diffeology with the set of n-balls $\{B^n_j\,;\, j\in J_n\}$. Then for any plot $P : A \to X^{(n)}$, there is an open covering $\{A_\alpha\}$ of A, such that $P(A_\alpha)$ is in either $X^{(n-1)}$ or B^n_j for some $j \in J_n$ and $P_\alpha = P|_{A_\alpha}$ is a plot of $X^{(n-1)}$ or B^n_j, respectively. Let $U = X^{(n)} \smallsetminus X^{(n-1)}$ and $V = X^{(n)} \smallsetminus (\bigcup_{j\in J_n} \{\mathbf{0}_j\})$, where $\mathbf{0}_j \in B^n_j$ denotes the element corresponding to $\mathbf{0} \in B^n$.

Case $(\operatorname{Im} P_\alpha \subset X^{(n-1)})$**:** $P^{-1}_\alpha(U) = \emptyset$, and $P^{-1}_\alpha(V) = A_\alpha$.

Case $(\operatorname{Im} P_\alpha \subset B^n_j)$**:** $P^{-1}_\alpha(U) = P^{-1}_\alpha(\operatorname{Int} B^n_j)$, and $P^{-1}_\alpha(V) = P^{-1}_\alpha(B^n_j \smallsetminus \{\mathbf{0}_j\})$.

In each case, $P^{-1}_\alpha(U)$ and $P^{-1}_\alpha(V)$ are open in A_α and hence in A, which implies that $P^{-1}(U)$ and $P^{-1}(V)$ are open in A for any plot P. Thus U and V are open sets in $X^{(n)}$.

Similarly to the case when X is a topological CW complex, $\mathcal{U} = \{U, V\}$ is a nice open covering of $X^{(n)}$ with a normal partition of unity $\{\rho^U, \rho^V\}$, since λ is a smooth function. Then, similar arguments for a topological CW complex lead us to the following result.

Theorem A.2. *For a smooth CW complex X, there are natural isomorphisms*

$$H^q_{\mathcal{D}}(X) \cong H^q_{\mathcal{C}}(X) \cong H^q_{\square}(X) \cong H^q(X, \mathbb{R}) \cong \operatorname{Hom}(H_q(X), \mathbb{R}),$$

for any $q \geq 0$, and hence we have $H^1_{\mathcal{D}}(X) \cong H^1_{\mathcal{C}}(X) \cong H^1_{\square}(X) \overset{\rho}{\cong} \operatorname{Hom}(\pi_1(X), \mathbb{R})$.

Conjecture A.3. *For a smooth CW complex X, there are natural isomorphisms*

$$H^q_{\mathcal{D}_c}(X) \cong H^q_{\mathcal{C}_c}(X) \cong H^q_{\sqcup_c}(X), \quad \textit{for any} \quad q \geq 0.$$

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