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# LUSTERNIK-SCHNIRELMANN CATEGORY OF Spin(9)

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Abstract. We first give an upper bound of $\mathrm{cat}(E)$ the L-S category of a principal $G$-bundle $E$ for a connected compact group $G$ with a characteristic map $\alpha : \Sigma V \to G$. Assume that there is a cone-decomposition $\{F_i \,|\, 0 \leq i \leq m\}$ of $G$ in the sense of Ganea, that is compatible with multiplication. Then we have $\mathrm{cat}(E) \leq \mathrm{Max}(m{+}n, m{+}2)$ for $n \geq 1$, if $\alpha$ is compressible into $F_n \subseteq F_m \simeq G$ with trivial higher Hopf invariant $H_n(\alpha)$ (see Iwase [10]). Second, we introduce a new computable lower bound, $\mathrm{Mwgt}(X; \mathbb{F}_2)$, for $\mathrm{cat}(X)$. The two new estimates imply $\mathrm{cat}(\mathbf{Spin}(9)) = \mathrm{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 8 > 6 = \mathrm{wgt}(\mathbf{Spin}(9); \mathbb{F}_2)$, where $\mathrm{wgt}(-; R)$ is a category weight due to Rudyak and Strom.

## Introduction

In this paper, we work in the category of connected CW-complexes and continuous maps. The Lusternik-Schnirelmann category $\mathrm{cat}(X)$, L-S category for short, is the least integer $m$ such that there is a covering of $X$ by $(m{+}1)$ open subsets each of which is contractible in $X$. Ganea introduced a stronger notion of L-S category, $\mathrm{Cat}(X)$ the strong L-S category of $X$, which is equal to the cone-length by Ganea [4], that is, the least integer $m$ such that there is a set of cofibre sequences $\{A_i \to X_{i-1} \hookrightarrow X_i\}_{1 \leq i \leq m}$ with $X_0 = \{*\}$ and $X_m \simeq X$. Then by Ganea [4], we have $\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X){+}1$. Throughout this paper, we follow the notations in [12], which is based on [9, 10]: For a map $f : S^k \to X$, a homotopy set of higher Hopf invariants $H_m(f) = \{[H^\sigma_m(f)] \,|\, \sigma \text{ is a structure map of } \mathrm{cat}(X){\leq}m\}$ (or its stabilisation $\mathcal{H}_m(f) = \Sigma^\infty_* H_m(f))$ is referred simply as *a (stabilized) higher Hopf invariant* of $f$, which plays a crucial role in this paper.

A computable lower estimate is given by the classical cup-length. Here we give its definition in a slightly general fashion, which is inspired by the proof of $\mathrm{cat}(\mathbf{Sp}(2)) = 3$ given in [14]:



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**Definition 0.1** (I. [12]). (1) *Let $h$ be a multiplicative generalized cohomology.*

$$\mathrm{cup}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \forall\{v_0, \cdots, v_m \in \tilde{h}^*(X)\}\ v_0{\cdot}v_1{\cdot}\cdots{\cdot}v_m = 0\right\}.$$

(2) $$\mathrm{cup}(X) = \mathrm{Max}\left\{\mathrm{cup}(X;h) \,\middle|\, \begin{array}{l} h \textit{ is a multiplicative generalized} \\ \textit{cohomology} \end{array}\right\}.$$

Then we have $\mathrm{cup}(X;h) \leq \mathrm{cup}(X) \leq \mathrm{cat}(X)$ for any multiplicative generalized cohomology $h$. When $h$ is the ordinary cohomology with a coefficient ring $R$, we denote $\mathrm{cup}(X;h)$ by $\mathrm{cup}(X;R)$. This definition immediately implies the following.

**Remark 0.2.** $\mathrm{cup}(X) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \tilde{\Delta}^{m+1} : X \to \wedge^{m+1}X \textit{ is stably trivial}\right\}$.

Let $\{p^{\Omega X}_k : E^k(\Omega X) \to P^{k-1}(\Omega X)\,;\ k \geq 1\}$ be the $A_\infty$-structure of $\Omega X$ in the sense of Stasheff [19] (see also Iwase-Mimura [13] for some more properties). The relation between an $A_\infty$-structure and a L-S category gives the key observation in [9, 10, 11] to producing counter-examples to the Ganea conjecture on L-S category. On the other hand, Rudyak [18] and Strom [20] introduced a homotopy theoretical version of Fadell-Husseini's category weight (see [3]), which can be described as follows, for an element $u \in h^*(X)$ and a generalized cohomology $h$:

$$\mathrm{wgt}(u;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, (e^X_m)^*(u) \neq 0 \text{ in } h^*(P^m(\Omega X))\right\},$$

$$\mathrm{wgt}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \begin{array}{l} (e^X_m)^* : h^*(X) \to h^*(P^m(\Omega X)) \\ \text{is a monomorphism} \end{array}\right\},$$

where $e^X_m$ denotes the map $P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X$. Then we easily see

$$\mathrm{wgt}(X;h) = \mathrm{Max}\{\mathrm{wgt}(u;h) \,|\, u \in \tilde{h}^*(X)\}. \tag{0.1}$$

We remark that $\mathrm{wgt}(u;h) = s$ if and only if $u$ represents a non-zero class in $E^{s,*}_\infty$ of bar spectral sequence $\{(E^{*,*}_r, d^{*,*}_r) \,|\, r \geq 1\}$ converging to $h^*(X)$ with $E^{**}_2 \cong \mathrm{Ext}^{**}_{h_*(\Omega X)}(h_*, h_*)$. When $h$ is the ordinary cohomology with a coefficient ring $R$, we denote $\mathrm{wgt}(X;h)$ by $\mathrm{wgt}(X;R)$. In this paper, we introduce new computable invariants as follows:

**Definition 0.3.** *Let $h$ be a generalized cohomology: A homomorphism $\phi : h^*(X) \to h^*(Y)$ is called a $h$-morphism if it preserves the actions of all (unstable) cohomology operations on $h^*$.*

**Definition 0.4** (I. [12]). *Let $h$ be a generalized cohomology and $X$ a space. A module weight* $\mathrm{Mwgt}(X;h)$ *of $X$ with respect to $h$ is defined as follows:*

$$\mathrm{Mwgt}(X;h) = \mathrm{Min}\left\{m \geq 0 \,\middle|\, \begin{array}{l} \textit{There is an } h\textit{-morphism } \phi : h^*(P^m(\Omega X)) \to \\ h^*(X)\textit{, which is a left homotopy inverse of } (e^X_m)^*. \end{array}\right\}$$

When $h$ is the ordinary cohomology with coefficients in a ring $R$, we denote $\mathrm{Mwgt}(X;h)$ by $\mathrm{Mwgt}(X;R)$. These invariants satisfy the following inequalities:

$$\mathrm{cup}(X;R) \leq \mathrm{wgt}(X;R) \leq \mathrm{Mwgt}(X;R) \leq \mathrm{cat}(X).$$

Similar to the above definition of $\mathrm{cup}(X)$, we define the following invariants:

**Definition 0.5** (I. [12])**.** (1) $\mathrm{wgt}(X) = \mathrm{Max}\left\{\mathrm{wgt}(X;h) \,\middle|\, \begin{array}{l} h \textit{ is a generalized} \\ \textit{cohomology} \end{array}\right\}$
(2) $\mathrm{Mwgt}(X) = \mathrm{Max}\,\{\mathrm{Mwgt}(X;h) \,|\, h \textit{ is a generalized cohomology}\}$

**Remark 0.6.** *Let* $r\mathrm{cat}(-)$ *be Rudyak's stable L-S category, which is denoted as* $r(-)$ *in* [18]. *Then we have* $\mathrm{cup}(X) \leq \mathrm{wgt}(X) = r\mathrm{cat}(X) \leq \mathrm{Mwgt}(X) \leq \mathrm{cat}(X)$.

Let us denote by $Z^{(k)}$ the $k$-skeleton of a CW complex $Z$. To give an upper-bound for L-S category of the total space of a fibre bundle $F \hookrightarrow E \to B$, we need a refinement of results of Varadarajan [21] and Hardie [6], and corresponding result for strong category of Ganea [4]:

**Theorem 0.7** ([21, 6, 4])**.** (1) $\mathrm{cat}(E){+}1 \leq (\mathrm{cat}(F){+}1){\cdot}(\mathrm{cat}(B){+}1)$
(2) $\mathrm{Cat}(E){+}1 \leq (\mathrm{Cat}(F){+}1){\cdot}(\mathrm{Cat}(B){+}1)$.

In [16], Iwase-Mimura-Nishimoto gave such a refinement in the case when the base space $B$ is non-simply connected. But in this paper, we give another refinement in the case when the fibre bundle is a principal bundle over a double suspension space: Let $G$ be a compact Lie group with a cone-decomposition of length $m$:

$$(m \text{ cofibre sequences}) \quad K_i \xrightarrow{\rho_i} F_{i-1} \hookrightarrow F_i, \quad i \geq 1,$$

with $F_0 = \{*\}$ and $F_i = F_m \simeq G$, $i \geq m$. Then we obtain $\sigma_k : F_k \to P^k\Omega F_k$ for all $k \leq m$ as a right homotopy inverse of $e_k : P^k\Omega F_k \to P^\infty\Omega F_k \simeq F_k$ by induction on $k \geq 1$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
\{*\} & \hookrightarrow & F_1 & \hookrightarrow & \cdots & \hookrightarrow & F_m \\
\downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma_1} & & & & \downarrow{\scriptstyle \sigma_m} \\
\{*\} & \hookrightarrow & P^1\Omega F_1 & \hookrightarrow & \cdots & \hookrightarrow & P^m\Omega F_m \\
\downarrow{\scriptstyle e_0} & & \downarrow{\scriptstyle e_1} & & & & \downarrow{\scriptstyle e_m} \\
\{*\} & \hookrightarrow & F_1 & \hookrightarrow & \cdots & \hookrightarrow & F_m,
\end{array}$$

where $e_k{\circ}\sigma_k \sim 1_{F_k}$ for all $k \leq m$.

**Theorem 0.8.** *Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with characteristic map $\alpha : A{=}\Sigma V \to G$. Then we have $\mathrm{cat}(E) \leq \mathrm{Max}(m{+}n, m{+}2)$ for $n \geq 1$, if*

(1) *$\alpha$ is compressible into $F_n \subseteq F_m \simeq G$,*
(2) *$H^{\sigma_n}_n(\alpha) = 0$ and*
(3) *the restriction of the multiplication $\mu : G{\times}G \to G$ to $F_j{\times}F_n \subseteq F_m{\times}F_m \simeq G{\times}G$ is compressible into $F_{j+n} \subseteq F_m \simeq G$, $j{\geq}0$ as $\mu_{j,n} : F_j{\times}F_n \to F_{j+n}$ such that $\mu_{j,n}|_{F_{j-1}\times F_n} = \mu_{j-1,n}$.*

**Remark 0.9.** *If we choose $n = m{+}1$, then the assumptions (1) through (3) above are automatically satisfied. Thus we always have $\mathrm{Cat}(E) \leq 2\,\mathrm{Cat}(G){+}1$ which is a special case of Ganea's theorem (see Theorem 0.7 (2)).*

For **Spin**(9), we first observe the following result.

**Theorem 0.10.** $\mathrm{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) \geq 8 > 6 = \mathrm{wgt}(\mathbf{Spin}(9); \mathbb{F}_2)$.

These results imply the following result.

**Theorem 0.11.** $\mathrm{cat}(\mathbf{Spin}(9)) = \mathrm{Mwgt}(\mathbf{Spin}(9); \mathbb{F}_2) = 8$.

## 1. Proof of Theorem 0.8

From now on, we work in the homotopy category, and so we do not distinguish $G$ and $F_m$. Let $G \hookrightarrow E \to \Sigma^2 V$ be a principal bundle with characteristic map $\alpha : A{=}\Sigma V \to G$. The assumptions (1) and (3) in Theorem 0.8 allows us to construct a filtration $\{E_k\}_{0\leq k\leq n+m}$ of $E$: By using the James-Whitehead decomposition (see Theorem 1.15 of Whitehead [22]), we have

$$E = G \cup_\psi G{\times}CA, \quad \psi = \mu{\circ}(1_G{\times}\alpha) : G{\times}A \to G.$$

Firstly, we define two filtrations of $E$ as follows:

$$E_k = \begin{cases} F_k, & k \leq n, \\ F_k \cup_{\psi_{k-n-1}} F_{k-n-1}{\times}CA, & n < k \leq m{+}n, \end{cases}$$

$$E'_k = \begin{cases} F_k, & k < n, \\ F_k \cup_{\psi_{k-n}} F_{k-n}{\times}CA, & n \leq k \leq m{+}n, \end{cases}$$

where $\psi_j = \mu_{j,n}{\circ}(\alpha{\times}1) : F_j{\times}A \to F_{j+n}$ and $E = E'_{m+n}$ which immediately imply that $\mathrm{cat}(E) = \mathrm{cat}(E'_{m+n})$.

By using the assumption (3), we obtain the following cofibre sequences, similarly to the arguments given in [16]:

$$K_{k+1} \to E_k \hookrightarrow E_{k+1}, \text{ for } 0 \le k < n,$$

$$K_{k+1} \vee (K_{k-n} * A) \to E_k \hookrightarrow E_{k+1}, \text{ for } n \le k < m+n,$$

$$K_{k-n} * A \to E_k \hookrightarrow E'_k,$$

Similarly to the arguments given in [15, 16], we obtain

$$\operatorname{cat}(E_k) \le k \text{ and } \operatorname{cat}(E'_k) \le k+1 \quad \text{for any } k \ge n, \tag{1.1}$$

by induction on $k$. The following lemma can be deduced in a similar but easier manner to the main theorem of [11], using $H^{\sigma_n}_n(\alpha) = 0$, the assumption (2):

**Lemma 1.1.** $\operatorname{cat}(E'_{j+n}) \le j+n$ *for all* $j \ge 0$ *and* $n \ge 2$.

Lemma 1.1 and (1.1) imply $\operatorname{cat}(E) = \operatorname{cat}(E'_{m+n}) \le \operatorname{Max}(m+n, m+2)$, and hence we are left to show Lemma 1.1.

*Proof of Lemma 1.1.* We define a map $\hat{\psi}_j$ as follows:

$$\hat{\psi}_j = \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) : P^j \Omega F_j \times A \to P^{j+n} \Omega F_{j+n}.$$

Then we have $\hat{\psi}_j \circ (\sigma_j \times 1) = \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) \circ (\sigma_j \times 1) \sim \sigma_{j+n} \circ \mu_{j,n} \circ (1 \times \alpha) = \sigma_{j+n} \circ \psi_j$ and $e_{j+n} \circ \hat{\psi}_j = e_{j+n} \circ \sigma_{j+n} \circ \mu_{j,n} \circ (e_j \times \alpha) \sim \mu_{j,n} \circ (e_j \times \alpha) = \psi_j \circ (e_j \times 1)$. Thus the following diagram is commutative up to homotopy:

$$\begin{array}{ccccc}
F_j & \xleftarrow{\mathrm{pr}_1} & F_j \times A & \xrightarrow{\psi_j} & F_{j+n} \\
\downarrow{\scriptstyle \sigma_j} & & \downarrow{\scriptstyle \sigma_j \times 1} & & \downarrow{\scriptstyle \sigma_{j+n}} \\
P^j \Omega F_j & \xleftarrow{\mathrm{pr}_1} & P^j \Omega F_j \times A & \xrightarrow{\hat{\psi}_j} & P^{j+n} \Omega F_{j+n} \\
\downarrow{\scriptstyle e_j} & & \downarrow{\scriptstyle e_j \times 1} & & \downarrow{\scriptstyle e_{j+n}} \\
F_j & \xleftarrow{\mathrm{pr}_1} & F_j \times A & \xrightarrow{\psi_j} & F_{j+n}
\end{array} \tag{1.2}$$

Therefore, the space $E'_{j+n} = F_{j+n} \cup_{\psi_j} F_j \times CA$ is dominated by $P^{j+n} \Omega F_{j+n} \cup_{\hat{\psi}_j} P^j \Omega F_j \times CA$. Since $\alpha$ satisfies $H^{\sigma_n}_n(\alpha) = 0$, we have the following commutative diagram up to homotopy:

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & F_m \\
\downarrow{\scriptstyle \Sigma \operatorname{ad} \alpha} & & \downarrow{\scriptstyle \sigma_n} \\
\Sigma \Omega F_n & \hookrightarrow & P^n \Omega F_n,
\end{array}$$

where $\mathrm{ad}\,\alpha : V \to \Omega F_n$ is the adjoint map of $\alpha : A{=}\Sigma V \to F_n$. Thus $\sigma_n{\circ}\alpha$ is compressible into $\Sigma\Omega F_n$, and hence we have

$$\begin{aligned}\hat{\psi}_j &\sim \sigma_{j+n}{\circ}\mu_{j,n}{\circ}(1{\times}(e_n{\circ}\sigma_n)){\circ}(e_j{\times}\alpha) \sim \sigma_{j+n}{\circ}\mu_{j,n}{\circ}(e_j{\times}e_n){\circ}(1{\times}(\sigma_n{\circ}\alpha))\\ &\sim \sigma_{j+n}{\circ}\mu_{j,n}{\circ}(e_j{\times}e_n){\circ}(1{\times}\Sigma\,\mathrm{ad}\,\alpha) = \hat{\psi}'_j|_{P^i\Omega F_j\times\Sigma\Omega F_n}{\circ}(1{\times}\Sigma\,\mathrm{ad}\,\alpha),\end{aligned}$$

where $\hat{\psi}'_j = \sigma_{j+n}{\circ}\mu_{j,n}{\circ}(e_j{\times}e_n)$. Since $\mathrm{Cat}(P^i\Omega F_j{\times}\Sigma\Omega F_n) \le i{+}1$, we have that $\hat{\psi}'_j|_{P^i\Omega F_j\times\Sigma\Omega F_n}$ can be compressible into $P^{i+1}\Omega F_{j+n}$ for $i \le j$. This yields the following cone-decomposition of $P^{j+n}\Omega F_{j+n} \cup_{\hat{\psi}_j} P^j\Omega F_j{\times}CA$:

$$\Omega F_{j+n} \to \{*\} \hookrightarrow P^1\Omega F_{j+n},$$

$$E^2\Omega F_{j+n}{\vee}A \to P^1\Omega F_{j+n} \hookrightarrow P^2\Omega F_{j+n}{\cup}CA,$$

$$\begin{aligned}E^{i+2}\Omega F_{j+n}{\vee}E^i\Omega F_j{*}A \;&\to P^{i+1}\Omega F_{j+n}{\cup}P^{i-1}\Omega F_j{\times}CA\\ &\hookrightarrow P^{i+2}\Omega F_{j+n}{\cup}P^i\Omega F_j{\times}CA, \quad 0 < i \le j,\end{aligned}$$

$$\begin{aligned}E^{j+i}\Omega F_{j+n} \;&\to P^{j+i-1}\Omega F_{j+n}{\cup}P^j\Omega F_j{\times}CA\\ &\hookrightarrow P^{j+i}\Omega F_{j+n}{\cup}P^j\Omega F_j{\times}CA, \quad 2 < i \le n,\end{aligned}$$

for any $j \ge 0$ and $n \ge 2$. This implies $\mathrm{Cat}(P^{j+n}\Omega F_{j+n} \cup_{\hat{\psi}_j} P^j\Omega F_j{\times}CA) \le j{+}n$ for all $j \ge 0$ and $n \ge 2$, and hence $\mathrm{cat}(E'_{j+n}) = \mathrm{cat}(F_{j+n} \cup F_j{\times}CA) \le j{+}n$ for all $j \ge 0$ and $n \ge 2$. □

This completes the proof of Theorem 0.8.

## 2. Bar spectral sequence

To calculate our module weight $\mathrm{Mwgt}(X;\mathbb{F}_2)$ together with $\mathrm{wgt}(X;\mathbb{F}_2)$, we need to know the module structure of $H^*(P^m(\mathbf{Spin}(9));\mathbb{F}_2)$ over the Steenrod algebra modulo 2. By Borel [1], Bott [2], Ishitoya-Kono-Toda [7], Hamanaka-Kono [5] and Kono-Kozima [17], the following are known:

$$\begin{aligned}&H^*(\mathbf{Spin}(9);\mathbb{F}_2) = \mathbb{F}_2[x_3]/(x_3^4){\otimes}\wedge_{\mathbb{F}_2}(x_5, x_7, x_{15}),\\ &\qquad Sq^2x_3 = x_5,\ Sq^1x_5 = x_6, \quad x_i \in H^i(\mathbf{Spin}(9);\mathbb{F}_2),\end{aligned}$$

$$\begin{aligned}&H_*(\Omega\mathbf{Spin}(9);\mathbb{F}_2) = \wedge_{\mathbb{F}_2}(u_2){\otimes}\mathbb{F}_2[u_4, u_6, u_{10}, u_{14}],\\ &\qquad u_4Sq^2 = u_2,\ u_{10}Sq^2 = u_4^2,\ u_{14}Sq^4 = u_{10}, \quad u_{2i} \in H_{2i}(\Omega\mathbf{Spin}(9);\mathbb{F}_2),\end{aligned}$$

where we denote by $\wedge_R(a_{i_1}, \cdots, a_{i_t})$ the exterior algebra on $a_{i_1}, \cdots, a_{i_t}$ over $R$. We remark that the cohomology suspension of $x_{2i+1}$ is non-trivially given by $u_{2i}$

for $i = 1$, 2, 3 and 7. To determine $H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2)$, we have to study the bar spectral sequence $(E_r^{*,*}, d_r^{*,*})$ converging to $H^*(\mathbf{Spin}(9); \mathbb{F}_2)$:

$$E_1^{s,t} \cong \tilde{H}^{s+t}(P^s(\Omega\mathbf{Spin}(9)), P^{s-1}(\Omega\mathbf{Spin}(9)); \mathbb{F}_2) \cong \tilde{H}^t(\bigwedge^s \Omega\mathbf{Spin}(9); \mathbb{F}_2),$$

$$D_1^{s,t} \cong \tilde{H}^{s+t}(P^s(\Omega\mathbf{Spin}(9)); \mathbb{F}_2),$$

$$E_2^{*,*} \cong \mathrm{Ext}^{*,*}_{H_*(\Omega\mathbf{Spin}(9);\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[x_{1,2}]\otimes\wedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,10}, x_{1,14}),$$

$$E_\infty^{*,*} \cong \tilde{H}^*(\mathbf{Spin}(9); \mathbb{F}_2) \cong \mathbb{F}_2[x_{1,2}]/(x_{1,2}^4)\otimes\wedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,14}),$$

where $x_{1,2}$, $x_{1,4}$, $x_{1,6}$ and $x_{1,14}$ are permanent cycles by [17]. Therefore, there is only one differential $d_a(x_{1,10})$ $(a \geq 2)$ which is possibly non-trivial, and we have $E_a^{*,*} \cong E_2^{*,*}$ and $E_{a+1}^{*,*} \cong E_\infty^{*,*}$. Since $x_3$ is of height 4 in $H^*(\mathbf{Spin}(9); \mathbb{F}_2)$, we have $d_a(x_{1,10}) = x_{1,2}^4$, and hence $a = 3$. Thus we have the following:

$$d_r = 0 \text{ if } r \neq 3,\ d_3(x_{1,i}) = 0 \text{ if } i \neq 10,\ d_3(x_{1,10}) = x_{1,2}^4,$$

$$E_2^{*,*} \cong E_3^{*,*} \cong \mathbb{F}_2[x_{1,2}]\otimes\wedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,10}, x_{1,14}),$$

$$E_4^{*,*} \cong E_\infty^{*,*} \cong \mathbb{F}_2[x_{1,2}]/(x_{1,2}^4)\otimes\wedge_{\mathbb{F}_2}(x_{1,4}, x_{1,6}, x_{1,14}).$$

By truncating the above computations with the same differential $d_r$ to the spectral sequence for $P^m(\mathbf{Spin}(9))$ of Stasheff's type (similar to the computation in [8]), we are lead to the following proposition, and we leave the details to the reader.

**Proposition 2.1.** *Let $A = \mathbb{F}_2[x_3]/(x_3^4)\otimes\wedge_{\mathbb{F}_2}(x_5, x_7, x_{15})$. Then for $m \geq 0$, we have*

$$H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2) \cong \begin{cases} A^{[0]} \cong \mathbb{F}_2, & \text{if } m = 0, \\ A^{[m]} \oplus x_{11}\cdot A^{[m-1]} \oplus S_m, & \text{if } 3 \geq m \geq 1, \\ A^{[m]} \oplus x_{11}\cdot(A^{[m-1]}/A^{[m-4]}) \oplus S_m, & \text{if } m \geq 4 \end{cases}$$

*as modules, where $A^{[m]}$ $(m \geq 0)$ denotes the quotient module $A/D^{m+1}(A)$ of $A$ by the submodule $D^{m+1}(A) \subseteq A$ generated by all the products of $m{+}1$ elements in positive dimensions, $x_{11}\cdot(A^{[m-1]}/A^{[m-4]})$ $(m \geq 4)$ denotes a submodule corresponding to a submodule in $\mathbb{F}_2[x_3]/(x_3^4)\otimes\wedge_{\mathbb{F}_2}(x_5, x_7, x_{11}, x_{15})$ and $S_m$ satisfies $S_m\cdot\tilde{H}^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2) = 0$ and $S_m|_{P^{m-1}(\mathbf{Spin}(9))} = 0$.*

Some more comments might be required to the second direct summand of the above expressions of $H^*(P^m(\mathbf{Spin}(9)); \mathbb{F}_2)$, $m \geq 4$. The multiplication with $x_{11}$ is a fancy notation to describe the module basis and not a usual product. However, we may regard it is a *partial product* in the sense introduced in the next section.

## 3. Partial products

Since a diagonal map $\Delta_n^{\Omega X}=\Omega(\Delta_n^X):\Omega X\to\prod^n\Omega X=\Omega(\prod^n X)$ is a loop map, it induces a map of projective spaces:

$$P^m(\Delta_n^{\Omega X}):P^m(\Omega X)\to P^m(\Omega(\prod^n X)),$$

such that $e_m^{\prod^n X}\circ P^m(\Delta_n^{\Omega X})\sim\Delta_n^X\circ e_m^X$. As is seen in the proof of Theorem 1.1 in [9], there is a natural map

$$\begin{aligned}\varphi_m^X:P^m(\Omega(\prod^n X))\to\bigcup_{i_1+\cdots+i_n=m}&P^{i_1}(\Omega X)\times\cdots\times P^{i_n}(\Omega X)\\ &\subset P^m(\Omega X)\times\cdots\times P^m(\Omega X)\end{aligned}$$

such that $(e_m^X\times\cdots\times e_m^X)\circ\varphi_m^X=e_m^{\prod^n X}$. Let $\Delta_n^{X,m}=\varphi_m^X\circ P^m(\Delta_n^{\Omega X})$, which we call the $n$-th partial diagonal of $X$ of height $m$, or simply a *partial diagonal*

$$\begin{aligned}\Delta_n^{X,m}:P^m(\Omega X)\to\bigcup_{i_1+\cdots+i_n=m}&P^{i_1}(\Omega X)\times\cdots\times P^{i_n}(\Omega X)\\ &\subset P^m(\Omega X)\times\cdots\times P^m(\Omega X)\end{aligned}$$

such that $(e_m^X\times\cdots\times e_m^X)\circ\Delta_n^{X,m}\sim\Delta_n^X\circ e_m^X$. This partial diagonal also yields the reduced version

$$\begin{aligned}\overline{\Delta}_n^{X,m}:P^m(\Omega X)\to\bigcup_{i_1+\cdots+i_n=m}&P^{i_1}(\Omega X)\wedge\cdots\wedge P^{i_n}(\Omega X)\\ &\subset P^{m-n+1}(\Omega X)\wedge\cdots\wedge P^{m-n+1}(\Omega X)\end{aligned}$$

such that $(e_{m-n+1}^X\wedge\cdots\wedge e_{m-n+1}^X)\circ\overline{\Delta}_n^{X,m}\sim\overline{\Delta}_n^X\circ e_m^X$, where $\overline{\Delta}_n^X:X\to\bigwedge^n X$ denotes the reduced diagonal. Let us call $\overline{\Delta}_n^{X,m}$ the $n$-th reduced partial diagonal of $X$ of height $m$, or simply a *reduced partial diagonal*.

As is well-known, the product of a multiplicative generalized cohomology $h$ is given by (reduced) diagonal, i.e.,

$$v_1\cdot\cdots\cdot v_n=(\overline{\Delta}_n^X)^*(v_1\otimes\cdots\otimes v_n)\in\bar{h}^*(X),\quad\text{for any } v_1,\cdots,v_n\in\bar{h}^*(X),$$

where $\bar{h}$ denotes the reduced cohomology associated with $h$. So it is natural to define a 'partial' product as the following way:

**Definition 3.1.** *For any elements $v_1,\cdots,v_n\in\bar{H}^*(\Sigma\Omega X;\mathbb{F}_2)$ which are restrictions of elements in $\bar{H}^*(P^{m-n+1}(\Omega X);\mathbb{F}_2)$, we define a partial product $v_1\cdot\cdots\cdot v_n=(\overline{\Delta}_n^{\Omega X,m})^*(v_1\otimes\cdots\otimes v_n)$ in $\bar{H}^*(P^m(\Omega X);\mathbb{F}_2)$.*

**Remark 3.2.** *Since* $x_{11}$ *can be extended to an element in* $\bar{H}^*(P^3(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$*, we have partial products* $x_{11}{\cdot}v_1{\cdot}\cdots{\cdot}v_{n-1} = (\overline{\Delta}^{\mathbf{Spin}(9),m}_{n-1})^*(x_{11}{\otimes}v_1{\otimes}\cdots{\otimes}v_{n-1})$ *for any elements* $v_1,\cdots,v_{n-1} \in \bar{H}^*(P^3(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$*,* $m{-}2 \le n \le m$*. In the direct sum decomposition of* $H^*(P^m(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$ *given in Proposition 2.1, the direct summand* $x_{11}{\cdot}(A^{[m-1]}/A^{[m-4]})$ *is generated by such partial products.*

## 4. Proof of Theorem 0.10

We know $x_3^3x_5x_7x_{15}$ and $x_{11}{\cdot}x_3^3x_5x_7$ exist non-trivially in $H^*(P^8(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$ but $x_{11}{\cdot}x_3^3x_5x_7$ does not exist in $H^*(P^9(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$ by Proposition 2.1. To observe what happens on the element $x_{11}{\cdot}x_3^3x_5x_7$ in $H^*(P^8(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$, we must recall the bar spectral sequence $(E_r^{*,*}, d_r^{*,*})$:

$$[(p_9^{\Omega\mathbf{Spin}(9)})^*(x_{11}{\cdot}x_3^3x_5x_7)] = d_3(x_{1,2}^3x_{1,4}x_{1,6}x_{1,10}) = \pm x_{1,2}^7x_{1,4}x_{1,6} \neq 0 \text{ in } E_3^{*,*},$$

where we denote by $[\beta]$ the corresponding class in $E_3^{*,*}$ to an element $\beta \in E_1^{*,*}$. Thus $(p_9^{\Omega\mathbf{Spin}(9)})^*(x_3^3x_5x_7x_{11}) \neq 0$ in $E_1^{9,*} = \tilde{H}^*(\bigwedge^9 \Omega\mathbf{Spin}(9);\mathbb{F}_2)$, and hence $x_{11}{\cdot}x_3^3x_5x_7$ does not exist in $\tilde{H}^*(P^9(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$, but does in $\tilde{H}^*(P^8(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$.

By [17], we know $Sq^4(x_{11}) = x_{15}$ in $H^*(P^1(\mathbf{Spin}(9));\mathbb{F}_2)$, and hence $Sq^4(x_{11}) = x_{15}$ modulo $S_3$ in $H^*(P^3(\mathbf{Spin}(9));\mathbb{F}_2)$ for dimensional reasons. Thus we have

$$Sq^4(x_{11}{\cdot}x_3^3x_5x_7) = x_3^3x_5x_7x_{15}, \quad \text{in } H^*(P^7(\Omega\mathbf{Spin}(9));\mathbb{F}_2). \tag{4.1}$$

$$Sq^4(x_{11}{\cdot}x_3^3x_5x_7) = x_3^3x_5x_7x_{15} + w, \quad w \in S_8 \text{ in } H^*(P^8(\Omega\mathbf{Spin}(9));\mathbb{F}_2). \tag{4.2}$$

The equation (4.1) implies that any left inverse epimorphism of $(e_7^{\mathbf{Spin}(9)})^*$

$$\phi : H^*(P^7(\Omega\mathbf{Spin}(9));\mathbb{F}_2) \to H^*(\mathbf{Spin}(9);\mathbb{F}_2)$$

does not preserve the action of the modulo 2 Steenrod operations: if such a epimorphism $\phi$ did preserve the action of the modulo 2 Steenrod operations, the element $\phi(x_3^3x_5x_7x_{15}) = x_3^3x_5x_7x_{15}$ in $H^*(\mathbf{Spin}(9);\mathbb{F}_2)$ should lie in the image of $Sq^4$, since $x_3^3x_5x_7x_{15}$ lies in the image of $Sq^4$ in $H^*(P^7(\Omega\mathbf{Spin}(9));\mathbb{F}_2)$ by (4.1). It contradicts to the fact that $H^{32}(\mathbf{Spin}(9);\mathbb{F}_2) = 0$. Thus we have $\mathrm{Mwgt}(\mathbf{Spin}(9);\mathbb{F}_2) \ge 8$.

On the other hand, we can easily see that each generator of $H^*(\mathbf{Spin}(9);\mathbb{F}_2) \cong \mathbb{F}_2[x_3]/(x_3^4){\otimes}\wedge_{\mathbb{F}_2}(x_5,x_7,x_{15})$ has category weight 1, and hence by (0.1), we have $\mathrm{wgt}(\mathbf{Spin}(9);\mathbb{F}_2) = 6$. This completes the proof of Theorem 0.10.

## 5. Proof of Theorem 0.11

By [15], we can easily see that $\mathbf{Spin}(7)$ admits a cone-decomposition which satisfies the condition 3 in Theorem 0.8. Since $x_{15} \in H^{15}(\mathbf{Spin}(7));\mathbb{F}_2)$ is the modulo 2 reduction of a generator of $H^{15}(\mathbf{Spin}(7);\mathbb{Z}) \cong \mathbb{Z}$, the image of the attaching map $\alpha$ of the 15-cell corresponding to $x_{15}$ must lie in $\mathbf{Spin}(7)^{(13)}$ the 13-skeleton of $\mathbf{Spin}(7)$, where $\mathbf{Spin}(7)^{(13)}$ is contained in $F_3(\mathbf{Spin}(7))$. To observe that the attaching map $\alpha$ satisfies the condition of Theorem 0.8 with $n = 3$, we need to show that $H_3^{\sigma_3}(\alpha) = 0$. Then we obtain $\mathrm{cat}(\mathbf{Spin}(9)) \leq \mathrm{Cat}(\mathbf{Spin}(7)) + n = 5+3 = 8$ by Theorem 0.8, while we know $\mathrm{Mwgt}(\mathbf{Spin}(9);\mathbb{F}_2) \geq 8$ by Theorem 0.10, and hence

$$\mathrm{cat}(\mathbf{Spin}(9)) = \mathrm{Mwgt}(\mathbf{Spin}(9);\mathbb{F}_2) = 8.$$

Let $\sigma_3 : F_3(\mathbf{Spin}(7)) \to P^3(\Omega F_3(\mathbf{Spin}(7)))$ be the canonical structure map of $\mathrm{cat}(F_3(\mathbf{Spin}(7))) = 3$. Then we are left to show that $H_3^{\sigma_3}(\alpha) = 0$. By definition,

$$H_3^{\sigma_3}(\alpha) : S^{14} \to E^4(\Omega F_3(\mathbf{Spin}(7))),$$

where $F_3(\mathbf{Spin}(7)) = \mathbf{G}_2^{(11)} \cup_{\Sigma\mathbb{C}P^2} \Sigma\mathbb{C}P^3 \cup (\text{higher cells} \geq 8)$. Since $\Omega\mathbf{G}_2^{(11)}$ has the homotopy type of $\mathbb{C}P^2 \cup (\text{higher cells} \geq 6)$, we know $\Omega F_3(\mathbf{Spin}(7))$ has the homotopy type of $\mathbb{C}P^3 \cup (\text{higher cells} \geq 6)$. Thus we observe that $E^4(\Omega F_3(\mathbf{Spin}(7)))$ has the homotopy type of

$$\Sigma^3\mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3\mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \cup (\text{higher cells} \geq 15).$$

It is well-known that $\Sigma\mathbb{C}P^3 = \Sigma\mathbb{C}P^2 \cup_{\omega_3} e^7$, $\omega_3 : S^6 \to S^3 \subset \Sigma\mathbb{C}P^3$, and hence we have $\Sigma^3\mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 = \Sigma^3\mathbb{C}P^2 \wedge S^2 \wedge S^2 \wedge S^2 \cup_{2\nu_{11}} e^{15}$, since $\omega_n = 2\nu_n$ for $n \geq 5$. An easy computation on the cohomology groups shows that $\mathbb{C}P^2 \wedge \mathbb{C}P^2$ has the homotopy type of $(\Sigma^2\mathbb{C}P^2 \vee S^6) \cup_\beta e^8$, $\beta : S^7 \xrightarrow{\mu} S^7 \vee S^7 \xrightarrow{3\nu_4 \vee \eta} S^4 \vee S^6 \subset \Sigma^2\mathbb{C}P^2 \vee S^6$, where $\mu$ denotes the unique co-Hopf structure of $S^7$. Then we obtain, up to higher cells in dimension $\geq 10$, that $[(\Sigma^2\mathbb{C}P^2 \vee S^6) \cup_\beta e^8] \wedge \mathbb{C}P^2 = (\Sigma^2\mathbb{C}P^2 \wedge \mathbb{C}P^2 \vee \Sigma^6\mathbb{C}P^2) \cup_{\Sigma^2\beta} e^{10} = (\Sigma^2\mathbb{C}P^2 \wedge \mathbb{C}P^2 \cup_{3\nu_6} e^{10}) \vee \Sigma^6\mathbb{C}P^2 = (\Sigma^4\mathbb{C}P^2 \cup_{3\nu_6} e^{10} \vee S^8) \cup_{\Sigma^2\beta} e^{10} \vee \Sigma^6\mathbb{C}P^2 = \Sigma^4\mathbb{C}P^2 \cup_{3\nu_6} e^{10} \vee \Sigma^6\mathbb{C}P^2 \vee \Sigma^6\mathbb{C}P^2$. Hence we have, up to higher cells in dimension $\geq 12$, that $[(\Sigma^2\mathbb{C}P^2 \vee S^6) \cup_\beta e^8](\wedge \Sigma^2\mathbb{C}P^2 \vee S^6) = (\Sigma^6\mathbb{C}P^2 \cup_{3\nu_8} e^{12}) \vee \Sigma^8\mathbb{C}P^2 \vee \Sigma^8\mathbb{C}P^2 \vee \Sigma^8\mathbb{C}P^2$. Thus we obtain that $E^4(\Omega F_3(\mathbf{Spin}(7))) = \Sigma^3\mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3\mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2 \wedge \mathbb{C}P^2$ has the homotopy type of

$$\Sigma^3\mathbb{C}P^3 \wedge S^2 \wedge S^2 \wedge S^2 \cup \Sigma^3[(\Sigma^2\mathbb{C}P^2 \vee S^6) \cup_\beta e^8] \wedge (\Sigma^2\mathbb{C}P^2 \vee S^6)$$
$$\cup (\Sigma^2\mathbb{C}P^2 \vee S^6) \wedge [(\Sigma^2\mathbb{C}P^2 \vee S^6) \cup_\beta e^8] \cup (\text{higher cells} \geq 15),$$
$$= (\Sigma^9\mathbb{C}P^2 \cup_{3\nu_{11}} e^{15} \cup_{2\nu_{11}} e^{15})$$
$$\vee \Sigma^{11}\mathbb{C}P^2 \vee \Sigma^{11}\mathbb{C}P^2 \vee \Sigma^{11}\mathbb{C}P^2 \cup (\text{higher cells} \geq 15)$$
$$= (\Sigma^9\mathbb{C}P^2 \cup_{\nu_{11}} e^{15}) \vee \Sigma^{11}\mathbb{C}P^2 \vee \Sigma^{11}\mathbb{C}P^2 \vee \Sigma^{11}\mathbb{C}P^2 \cup (\text{higher cells} \geq 15).$$

Then an elementary computation shows that $\pi_{14}(E^4(\Omega F_3(\mathbf{Spin}(7)))) = 0$, and hence $H_3^{\sigma_3}(\alpha) = 0$. This completes the proof of Theorem 0.11.

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