Lusternik-Schnirelmann category of non-simply connected compact simple Lie groups

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Abstract

Let $F \hookrightarrow X \to B$ be a fibre bundle with structure group G, where B is $(d{-}1)$-connected and of finite dimension, $d \geq 1$. We prove that the strong L-S category of X is less than or equal to $m+\frac{\dim B}{d}$, if F has a cone decomposition of length m under a compatibility condition with the action of G on F. This gives a consistent prospect to determine the L-S category of non-simply connected Lie groups. For example, we obtain $\mathrm{cat}\,(PU(n)) \leq 3(n{-}1)$ for all $n \geq 1$, which might be best possible, since we have $\mathrm{cat}\,(\mathrm{PU}(p^r)) = 3(p^r{-}1)$ for any prime p and $r \geq 1$. Similarly, we obtain the L-S category of $\mathrm{SO}(n)$ for $n \leq 9$ and $\mathrm{PO}(8)$. We remark that all the above Lie groups satisfy the Ganea conjecture on L-S category.

Key words: Lusternik-Schnirelmann category; cone decomposition; Lie group; Ganea conjecture
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1 Introduction

The Lusternik-Schnirelmann category $\operatorname{cat}(X)$, L-S category for short, is the least integer m such that there is a covering of X by $(m+1)$ open subsets each of which is contractible in X.

Ganea [5] introduced a stronger notion of L-S category, $\operatorname{Cat}(X)$, which is equal to the cone-length, that is, the least integer m such that there is a set of cofibre sequences $\{A_i \to X_{i-1} \hookrightarrow X_i\}_{1\leq i\leq m}$ with $X_0 = \{*\}$ and X_m homotopy equivalent to X.

The weak L-S category $w\mathrm{cat}(X)$ is the least integer m such that the reduced diagonal map $\bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X$ is trivial where $\bigwedge^{m+1} X$ is the smash product. The stabilised version of the invariant $w\mathrm{cat}(X)$ is given as the least integer m such that the reduced diagonal map $\bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X$ is *stably* trivial. Let us denote it by $\mathrm{cup}(X)$, the *cup-length* of X.

In 1971, Ganea [6] posed 15 problems on L-S category and its related topics: Computation of L-S category for various manifolds is given as the first problem and the second problem is known as the Ganea conjecture on L-S category. These problems especially the first two problems have attracted many authors such as James and Singhof [15], [28], [25], [26], [27], [16], Gómez-Larrañaga and González-Acuña [7], Montejano [18], Oprea and Rudyak [20], [21], [19] and the authors [10], [11], [12], [13], [14]. In [11,12], the first author gave a counter example as a manifold to the Ganea conjecture on L-S category.

Especially for L-S category of compact connected simple Lie groups, the followings have already been known:

$$\begin{aligned}
&\operatorname{cat}(\mathrm{Sp}(1)) = \operatorname{cat}(\mathrm{SU}(2)) = \operatorname{cat}(\mathrm{Spin}(3)) = 1,\\
&\operatorname{cat}(\mathrm{SU}(3)) = 2, \qquad \operatorname{cat}(\mathrm{SO}(3)) = 3,
\end{aligned}$$

since $\mathrm{Sp}(1) = \mathrm{SU}(2) = \mathrm{Spin}(3) = S^3$, $\mathrm{SU}(3) = \Sigma\mathbb{C}P^2 \cup e^8$ and $\mathrm{SO}(3) = \mathbb{R}P^3$. Schweitzer [24] showed

$$\operatorname{cat}(\mathrm{Sp}(2)) = 3$$

using functional cohomology operations. Singhof [25,27] showed

$$\begin{aligned}
&\operatorname{cat}(\mathrm{SU}(n)) = n-1,\\
&\operatorname{cat}(\mathrm{Sp}(n)) \geq n+1, \quad \text{if } n \geq 2.
\end{aligned}$$

Also we know

$$\operatorname{cat}(\mathrm{G}_2) = 4$$

by [15] (see [13]). James and Singhof [16] showed

$$\operatorname{cat}(\mathrm{SO}(5)) = 8.$$

The first and second authors [13] and Fernández-Suárez, Gómez-Tato, Strom and Tanré [4] proved

$$\operatorname{cat}(\mathrm{Sp}(3)) = 5,$$
$$\operatorname{cat}(\mathrm{Sp}(n)) \geq n+2 \quad \text{if } n \geq 3,$$

by showing the reduced diagonal $\bar{\Delta}^5$ is given by the Toda bracket $\{\eta, \nu, \eta\} = \nu^2$. The authors [14] showed

$$\operatorname{cat}(\mathrm{Spin}(7)) = 5, \quad \operatorname{cat}(\mathrm{Spin}(8)) = 6$$

using explicit cone decompositions of Spin(7) and SU(4). Then the Ganea conjecture on L-S category holds for all these Lie groups, since the L-S and the strong L-S categories are equal to the cup-length:

Fact 1.1 *If* $\operatorname{cat}(X) = \operatorname{cup} X$*, then the Ganea conjecture on L-S category holds for* X*, i.e.,* $\operatorname{cat}(X \times S^n) = \operatorname{cat}(X) + 1$ *for all* $n \geq 1$.

In fact, we have $\operatorname{cup}(X \times S^n) = \operatorname{cup}(X) + 1$ in general.

For any multiplicative cohomology theory h, we define $\operatorname{cup}(X; h)$, the *cup-length* with respect to h, by the least integer m such that $u_0 \cdot \cdots \cdot u_m = 0$ for any $m+1$ elements $u_i \in \tilde{h}^*(X)$. When h is the ordinary cohomology theory with coefficient ring R, $\operatorname{cup}(X; h)$ is often denoted as $\operatorname{cup}(X; R)$.

Theorem 1.2 *For any CW-complex* X *we have*

$$\operatorname{cup}(X) = \max\{\operatorname{cup}(X; h) \mid h \textit{ is any multiplicative cohomology theory}\}.$$

Proof. It is easy to see that $\operatorname{cup}(X) \geq \operatorname{cup}(X; h)$, and hence we have $\operatorname{cup}(X) \geq \max\{\operatorname{cup}(X; h) \mid h \text{ is any multiplicative cohomology theory}\}$. Thus we must show

$$\operatorname{cup}(X) \leq \max\{\operatorname{cup}(X; h) \mid h \text{ is any multiplicative cohomology theory}\}.$$

Let $m = \max\{\operatorname{cup}(X; h) \mid h \text{ is any multiplicative cohomology theory}\}$ and h_X be the multiplicative cohomology theory represented by the following wedge sum of iterated smash products of suspension spectrum $\Sigma^\infty X$:

$$S^0 \vee \Sigma^\infty X \vee \Sigma^\infty {\textstyle\bigwedge}^2 X \vee \cdots \vee \Sigma^\infty {\textstyle\bigwedge}^i X \vee \cdots .$$

Let $\iota \in \tilde{h}^*_X(X)$ be the element which is represented by the inclusion map into the second factor $\Sigma^\infty X$ of the above wedge sum. Then by the definition of the cup-length, we have $\iota^{m+1} = 0$ which is represented by the reduced diagonal map $\bar{\Delta}^{m+1} : X \to \bigwedge^{m+1} X$ in the $(m+2)$-nd factor $\Sigma^\infty \bigwedge^{m+1} X$ of the above wedge sum. Hence we have $\operatorname{cup}(X) \leq m$ the desired inequality. Thus we obtain the result. □

Let $P^m(\Omega X)$ be the m-th projective space, in the sense of Stasheff [29], such that there is a homotopy equivalence $P^\infty(\Omega X) \simeq X$. The following theorem is obtained by Ganea (see also [10] and Sakai [23]).

Theorem 1.3 (Ganea [5]) *$\mathrm{cat}\,(X) \le m$ if and only if there is a map $\sigma : X \to P^m(\Omega X)$ such that $e^X_m \circ \sigma \sim 1_X$, where $e^X_m : P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X$.*

Using this, Rudyak [21,22] introduced a stable L-S category, $r\mathrm{cat}\,(X)$, which is the least integer m such that there is a stable map $\sigma : X \to P^m(\Omega X)$ satisfying $e^X_m \circ \sigma \sim 1_X$, another stabilised version of L-S category.

Rudyak [20] [21] and Strom [30] introduced the following invariant to calculate $r\mathrm{cat}\,(X)$: Let $\mathrm{wgt}(X; h)$ be the least integer m such that the homomorphism $(e^X_m)^* : \tilde{h}^*(X) \to \tilde{h}^*(P^m(\Omega X))$ is injective for any cohomology theory h. When h is the ordinary cohomology theory with coefficient ring R, $\mathrm{wgt}(X; h)$ is often denoted as $\mathrm{wgt}(X; R)$.

Since a product of any $m+1$ elements of $\tilde{h}^*(P^m(\Omega X))$ is trivial, we have $\mathrm{cup}(X; h) \le \mathrm{wgt}(X; h)$ for any multiplicative cohomology theory h. Hence we have $\mathrm{cup}(X) \le \mathrm{wgt}(X)$, where we denote $\mathrm{wgt}(X) = \max\{\mathrm{wgt}(X; h) \mid h \text{ is any cohomology theory}\}$.

Remark 1.4 *For any ring R, we know $\mathrm{cup}(\mathrm{Sp}(2); R) = \mathrm{wgt}(\mathrm{Sp}(2); R) = 2 < 3 = \mathrm{cat}\,(\mathrm{Sp}(2))$. But an easy calculation of algebra structure of $KO^*(\mathrm{Sp}(2))$ yields $\mathrm{cup}(\mathrm{Sp}(2); KO) = \mathrm{wgt}(\mathrm{Sp}(2); KO) = 3 = \mathrm{cat}\,(\mathrm{Sp}(2))$.*

The following theorem is due to Rudyak [21,22], although we do not know the precise relation between $w\mathrm{cat}\,(X)$ and $r\mathrm{cat}\,(X)$.

Theorem 1.5 *For any CW complex X, we have*

$$r\mathrm{cat}\,(X) = \mathrm{wgt}\,X$$

and hence we have the following relations among categories:

$$\mathrm{cup}(X) \le w\mathrm{cat}\,(X), r\mathrm{cat}\,(X) \le \mathrm{cat}\,(X) \le \mathrm{Cat}\,(X).$$

Using this stabilised version of L-S category, we have the following theorem.

Theorem 1.6 (Rudyak [21,22]) *If $\mathrm{cat}\,(X) = r\mathrm{cat}\,(X)$, then the Ganea conjecture on L-S category holds for X.*

In fact, we have $r\mathrm{cat}\,(X \times S^n) = r\mathrm{cat}\,(X){+}1$ in general ([21,22]).

2 Main results

From now on, we work in the category of connected CW-complexes and continuous maps. We denote by $Z^{(k)}$ the k-skeleton of a CW complex Z.

Theorem 2.1 (James [15],Ganea [5]) *Let X be a $(d-1)$-connected space of finite dimension. Then $\mathrm{cat}\,(X) \leq \mathrm{Cat}\,(X) \leq [\frac{\dim(X)}{d}]$, where $[a]$ denotes the biggest integer $\leq a$.*

In this paper, we extend this for a total space of a fibre bundle, to determine L-S categories of SO(n) for $n \leq 9$, PO(8) and PU(p^r) (and the other quotient groups of SU(p^r)), which also gives an alternative proof of a result due to James and Singhof [16] on SO(5).

We assume that B is a $(d{-}1)$-connected finite dimensional CW complex ($d \geq 1$), whose cells are concentrated in dimensions $0, 1, \cdots, s \mod d$ for some s, $(0{\leq}s{\leq}d{-}1)$. Let $F \hookrightarrow X \to B$ be a fibre bundle with structure group G, a compact Lie group. Then we have the associated principal bundle $G \hookrightarrow E \stackrel{\pi}{\to} B$ with G-action $\psi : G{\times}F \to F$ on F and hence $X = E{\times}_G F$.

Let $K_i \stackrel{\rho_i}{\to} F_{i-1} \hookrightarrow F_i$, $(1{\leq}i{\leq}m)$ be m cofibre sequences with $F_0 = \{*\}$ and F_m homotopy equivalent to F. We consider the following compatibility condition of the above cone decomposition of F and the action of G on F.

Assumption 1 *$\psi|_{G^{(d\cdot(i+1)+s-1)}\times F_j} : G^{(d\cdot(i+1)+s-1)}{\times}F_j \to F$ is compressible into F_{i+j}, $0 \leq i, j \leq i{+}j \leq m$.*

Remark 2.2 *(1) Let $F = G$ and $X = E$ be the total space of a principal bundle over a path-connected space B and $d = 1$. Then any cone decomposition of F such that $F_i = F^{(n_i)}$ with $0 < n_1 < n_2 < \cdots < n_m = \dim(F)$ satisfies Assumption 1 with $s = 0$.*

(2) Let $F \hookrightarrow X \to B$ be a trivial bundle. Then any cone decomposition of F satisfies the compatibility Assumption 1 with $s = d{-}1$.

Our main result is stated as follows:

Theorem 2.3 *Let B be a $(d{-}1)$-connected finite dimensional CW complex ($d \geq 1$), whose cells are concentrated in dimensions $0, 1, \cdots, s \mod d$ for some s, $0 \leq s \leq d{-}1$. Let $F \hookrightarrow X \to B$ be a fibre bundle with fibre F whose structure group is a compact Lie group G. If F has a cone decomposition with the compatibility Assumption 1 for d, then $\mathrm{Cat}\,(X) \leq m + [\frac{\dim B}{d}]$.*

Corollary 2.4 *If F has a cone decomposition with the compatibility Assumption 1 for $s = d{-}1$ and also $m = \mathrm{Cat}\,(F)$, then $\mathrm{Cat}\,(X) \leq \mathrm{Cat}\,(F) + [\frac{\dim B}{d}]$.*

Remark 2.5 *Without Assumption 1, we only have*

$$\mathrm{Cat}\,(X){+}1 \le (\mathrm{Cat}\,(F){+}1){\cdot}(\mathrm{Cat}\,(B){+}1)$$

which is obtained immediately from the definition of Cat *by Ganea [5] and the corresponding results of Varadarajan [31] and Hardie [8] for* cat. *For example, the principal bundle* $\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(2) \to S^7$ *does satisfy Assumption 1 for* $d \le 3$*, but not if* $d \ge 4$*, and we have* $\mathrm{Cat}\,(\mathrm{Sp}(2)) \le \mathrm{Cat}\,(\mathrm{Sp}(1)){+}[\frac{7}{3}] = 3 > 2 = \mathrm{Cat}\,(\mathrm{Sp}(1)){+}[\frac{7}{4}]$. *In fact by Schweitzer [24], we know* $\mathrm{Cat}\,(\mathrm{Sp}(2)) = 3$.

Remark 2.6 *By Remark 2.2 (2), Theorem 2.3 generalises Theorem 2.1.*

By applying this, we first obtain the following general result:

Theorem 2.7 *Let* $C_m < SU(n)$ *be a central (cyclic) subgroup of order* m*. Then we have* $\mathrm{Cat}\,(\mathrm{SU}(n)/C_m) \le 3(n{-}1)$ *for all* $n \ge 1$.

This might be best possible, because we also obtain the following result.

Theorem 2.8 *We have*

$$\mathrm{Cat}\,(\mathrm{SU}(p^r)/C_{p^s}) = \mathrm{cat}\,(\mathrm{SU}(p^r)/C_{p^s}) = r\mathrm{cat}\,(\mathrm{SU}(p^r)/C_{p^s}) = 3(p^r{-}1)$$

where p *is a prime and* $1 \le s \le r$.

Similarly we obtain the following result.

Theorem 2.9 *We have*

$$\begin{aligned}
\mathrm{Cat}\,(\mathrm{SO}(6)) &= \mathrm{cat}\,(\mathrm{SO}(6)) = \mathrm{cup}(\mathrm{SO}(6)) = 9,\\
\mathrm{Cat}\,(\mathrm{SO}(7)) &= \mathrm{cat}\,(\mathrm{SO}(7)) = \mathrm{cup}(\mathrm{SO}(7)) = 11,\\
\mathrm{Cat}\,(\mathrm{SO}(8)) &= \mathrm{cat}\,(\mathrm{SO}(8)) = \mathrm{cup}(\mathrm{SO}(8)) = 12,\\
\mathrm{Cat}\,(\mathrm{SO}(9)) &= \mathrm{cat}\,(\mathrm{SO}(9)) = \mathrm{cup}(\mathrm{SO}(9)) = 20,\\
\mathrm{Cat}\,(\mathrm{PO}(8)) &= \mathrm{cat}\,(\mathrm{PO}(8)) = \mathrm{cup}(\mathrm{PO}(8)) = 18.
\end{aligned}$$

Remark 2.10 *Theorem 2.3 also provides an alternative proof for a result of James-Singhof [16], that is,* $\mathrm{Cat}\,(\mathrm{SO}(5)) = \mathrm{cat}\,(\mathrm{SO}(5)) = \mathrm{cup}(\mathrm{SO}(5)) = 8$ *(see Section 4).*

We summarise all the known cases in the following table, where each number given in the right hand side of a connected, compact, simple Lie group indicates its L-S category.

rank	1		2		3		4		n (≥ 5)	
A_n	SU(2)	1	SU(3)	2	SU(4)	3	SU(5)	4	SU(n+1)	n
					SO(6)	9			⋮	
	PU(2)	3	PU(3)	6	PU(4)	9	PU(5)	12	PU(n+1)	–
B_n	Spin(3)	1	Spin(5)	3	Spin(7)	5	Spin(9)	–	Spin(2n+1)	–
	SO(3)	3	SO(5)	8	SO(7)	11	SO(9)	20	SO(2n+1)	–
C_n	Sp(1)	1	Sp(2)	3	Sp(3)	5	Sp(4)	–	Sp(n)	–
	PSp(1)	3	PSp(2)	8	PSp(3)	–	PSp(4)	–	PSp(n)	–
D_n					Spin(6)	3	Spin(8)	6	Spin(2n)	–
					SO(6)	9	SO(8)	12	SO(2n)	–
					PO(6)	9	PO(8)	18	PO(2n)	–
									Ss(2n)	–
Except. types			G_2	4			F_4	–	E_6, E_7, E_8	–

where "-" indicates the unknown case.

Remark 2.11 *We recall that* $A_1 = B_1 = C_1$, $B_2 = C_2$ *and* $A_3 = D_3$, *and that the semi-spinor group* Ss(2n) *is defined only for* n *even.*

Taking into account the above table, we get the following by Theorem 1.6:

Corollary 2.12 *The Ganea conjecture on L-S category holds for every connected, compact, simple Lie group* G *when L-S category is known as above.*

The paper is organised as follows; In Section 3 we prove Theorem 2.3. In Section 4 we determine cat (SO(n)) for $n = 5, 6, 7, 8, 9$ and cat (PO(8)). In Section 5 we prove Theorem 2.7 and determine cat $(\mathrm{SU}(p^r)/C_{p^s})$.

3 Proof of Theorem 2.3

Let B_i be the $(d{\cdot}i{+}s)$-skeleton of B and $n{=}[\frac{\dim B}{d}]$ the biggest integer not exceeding $\frac{\dim B}{d}$. Then by Ganea [5], Theorem 2.1 implies that there are n cofibre sequences $A_i \xrightarrow{\lambda_i} B_{i-1} \hookrightarrow B_i$, $1{\leq}i{\leq}n$ with $B_0 = \{*\}$, $B_n = B$. Note that A_i is $(d{\cdot}i{-}2)$-connected and of dimension $(d{\cdot}i{+}s{-}1)$. Hence we obtain

$$B_i = B_{i-1} \cup_{\lambda_i} C(A_i), \quad \lambda_i : A_i \to B_{i-1}$$
$$A_i = A_i^{(d\cdot i+s-1)} = \textstyle\bigcup_{a=0}^{s} A_i^{(d\cdot i+a-1)}, \quad 1 \le i \le n,$$
$$B_0 = \{*\}, \quad B_n \simeq B.$$

Then there is a filtration of E by $E|_{B_i}$, $0 \le i \le n$, as follows (see Whitehead [32], for example):

$$E|_{B_i} = E|_{B_{i-1}} \cup_{\Lambda_i} C(A_i){\times}G, \quad \Lambda_i : A_i{\times}G \to E|_{B_{i-1}}, \quad 1 \le i \le n,$$
$$E|_{B_0} = \{*\}{\times}G, \quad E|_{B_n} \simeq E,$$

and $\tilde{\lambda}_i = \Lambda_i|_{A_i} : A_i \to E|_{B_{i-1}}$ gives a lift of $\lambda_i : A_i \to B_{i-1}$. Then by induction on i, we have

$$E|_{B_i} = \{*\}{\times}G \cup_{\Lambda_1} C(A_1){\times}G \cup_{\Lambda_2} \cdots \cup_{\Lambda_i} C(A_i){\times}G,$$
$$\Lambda_i : A_i{\times}G \xrightarrow{\tilde{\lambda}_i \times 1_G} E|_{B_{i-1}}{\times}G$$
$$= \Big(\{*\}{\times}G \cup_{\Lambda_1} C(A_1){\times}G \cdots \cup_{\Lambda_{i-1}} C(A_{i-1}){\times}G\Big){\times}G$$
$$\xrightarrow{1\times\mu} \{*\}{\times}G \cup_{\Lambda_1} C(A_1){\times}G \cdots \cup_{\Lambda_{i-1}} C(A_{i-1}){\times}G = E|_{B_{i-1}},$$

where μ is the multiplication of G. For dimensional reasons, we may regard

$$\tilde{\lambda}_i : (A_i, A_i^{(d\cdot i+a-1)}) \to (E^{(d\cdot i+s-1)}|_{B_{i-1}}, E^{(d\cdot i+a-1)}|_{B_{i-1}}), \quad 0 \le a \le s,$$

and $\mu(G^{(i)}{\times}G^{(j)}) \subset G^{(i+j)}$ up to homotopy. Then we have the following descriptions for all $k \ge d{\cdot}i{-}1$ and $j \ge d{-}1$:

$$E^{(k)}|_{B_i} = \left(\{*\}{\times}G \cup_{\Lambda_1} C(A_1){\times}G \cup_{\Lambda_2} \cdots \cup_{\Lambda_i} C(A_i){\times}G\right)^{(k)},$$
$$= \begin{pmatrix} \{*\}{\times}G^{(k)} \cup_{\Lambda_1} \bigcup_{\ell=0}^{s}(C(A_1^{(d+\ell-1)}){\times}G^{(k-d-\ell)}) \\ \cdots \cup_{\Lambda_i} \bigcup_{\ell=0}^{s}(C(A_i^{(d\cdot i+\ell-1)}){\times}G^{(k-d\cdot i-\ell)}) \end{pmatrix},$$

$$\Lambda_i : A_i^{(d\cdot i+\ell-1)}{\times}G^{(j-\ell)} \xrightarrow{\tilde{\lambda}_i \times 1_{G^{(j)}}} E^{(d\cdot i+\ell-1)}|_{B_{i-1}}{\times}G^{(j-\ell)}$$
$$= \begin{pmatrix} \{*\}{\times}G^{(d\cdot i+\ell-1)} \\ \cup_{\Lambda_1} \bigcup_{a=0}^{s}(C(A_1^{(d+a-1)}){\times}G^{(d\cdot(i-1)+\ell-a-1)}) \\ \cdots \cup_{\Lambda_{i-1}} \bigcup_{a=0}^{s}(C(A_{i-1}^{(d\cdot(i-1)+a-1)}){\times}G^{(d+\ell-a-1)}) \end{pmatrix}{\times}G^{(j-\ell)}$$
$$\xrightarrow{1\times\mu} \begin{pmatrix} \{*\}{\times}G^{(d\cdot i+j-1)} \cup_{\Lambda_1} \bigcup_{a=0}^{s}(C(A_1^{(d+a-1)}){\times}G^{(d\cdot(i-1)+j-a-1)}) \\ \cdots \cup_{\Lambda_{i-1}} \bigcup_{a=0}^{s}(C(A_{i-1}^{(d\cdot(i-1)+a-1)}){\times}G^{(d+j-a-1)}) \end{pmatrix}$$

$$= \Big(\{*\}{\times}G \cup_{\Lambda_1} C(A_1){\times}G \cup_{\Lambda_2} \cdots \cup_{\Lambda_{i-1}} C(A_{i-1}){\times}G\Big)^{(d\cdot i+j-1)}$$
$$= E^{(d\cdot i+j-1)}|_{B_{i-1}}.$$

Similarly, we obtain the following filtration $\{E'_k\}_{0\le k\le n+m}$ of $E{\times}_G F$.

$$E'_k = \begin{cases} F_k \cup_{\Lambda'_1} C(A_1){\times}F_{k-1} \cup_{\Lambda'_2} \cdots \cup_{\Lambda'_k} C(A_k){\times}F_0, & k \le n, \\ F_k \cup_{\Lambda'_1} C(A_1){\times}F_{k-1} \cup_{\Lambda'_2} \cdots \cup_{\Lambda'_n} C(A_n){\times}F_{k-n}, & n \le k, \end{cases}$$

$$\Lambda'_i : A_i{\times}F_j \xrightarrow{\tilde\lambda_i\times 1_{F_j}} E^{(d\cdot i+s-1)}|_{B_{i-1}}{\times}F_j$$
$$= \begin{pmatrix} G^{(d\cdot i+s-1)} \cup_{\Lambda_1} \bigcup_{a=0}^{s}(C(A_1^{(d+a-1)}){\times}G^{(d\cdot(i-1)+s-a-1)}) \\ \cdots \cup_{\Lambda_{i-1}} \bigcup_{a=0}^{s}(C(A_{i-1}^{(d\cdot(i-1)+a-1)}){\times}G^{(d+s-a-1)}) \end{pmatrix} {\times}F_j$$
$$\xrightarrow{1\times\psi} \begin{pmatrix} F_{i+j-1} \cup_{\Lambda'_1} \bigcup_{a=0}^{s}(C(A_1^{(d+a-1)}){\times}F_{i+j-2}) \\ \cdots \cup_{\Lambda'_{i-1}} \bigcup_{a=0}^{s}(C(A_{i-1}^{(d\cdot(i-1)+a-1)}){\times}F_j) \end{pmatrix}$$
$$= F_{i+j-1} \cup_{\Lambda'_1} C(A_1){\times}F_{i+j-2} \cdots \cup_{\Lambda'_{i-1}} C(A_{i-1}){\times}F_j$$
$$= E'_{i+j-1}|_{B_{i-1}},$$

since $\psi(G^{(d\cdot(\ell+1)+s-a-1)}{\times}F_j) \subseteq \psi(G^{(d\cdot(\ell+1)+s-1)}{\times}F_j) \subset F_{\ell+j}$ by Assumption 1. The above definition of Λ'_i also determines a map

$$\psi_{i,j} : E^{(d\cdot(i+1)+s-1)}|_{B_i}{\times}F_j \longrightarrow E'_{i+j}|_{B_i}$$

so that $\Lambda'_i = \psi_{i-1,j}\circ(\tilde\lambda_i{\times}1)$. Let us recall that $F_j = F_{j-1} \cup_{\rho_j} C(K_j)$ for $1 \le j \le m$. Then the definition of E'_k implies

$$E'_k = \begin{cases} E'_{k-1} \cup C(K_k) \cup C(A_1){\times}C(K_{k-1}) \cup \cdots \\ \qquad \cdots \cup C(A_{k-1}){\times}C(K_1) \cup C(A_k){\times}\{*\} & \text{for } k \le n, \\ E'_{k-1} \cup C(K_k) \cup C(A_1){\times}C(K_{k-1}) \cup \cdots \\ \qquad \cdots \cup C(A_{n-1}){\times}C(K_{k-n+1}) \cup C(A_n){\times}C(K_{k-n}) & \text{for } k > n. \end{cases}$$

To observe the relation between $\mathrm{Cat}\,(E'_{k-1})$ and $\mathrm{Cat}\,(E'_k)$, we introduce the following two relative homeomorphisms:

$$\chi(\rho_j) : (C(K_j), K_j) \to (F_{j-1} \cup C(K_j), F_{j-1}) \ (= (F_j, F_{j-1}))$$

$$\chi(\tilde{\lambda}_i) : (C(A_i), A_i) \to \ (E^{(d\cdot i+s-1)}|_{B_i} \cup C(A_i), E^{(d\cdot i+s-1)}|_{B_{i-1}}) \\ (\subset (E^{(d\cdot i+s)}|_{B_i}, E^{(d\cdot i+s-1)}|_{B_{i-1}})).$$

Then the attaching map of $C(A_i)\times C(K_j)$ is given by the Whitehead product $[\chi(\tilde{\lambda}_i), \chi(\rho_j)] : A_i * K_j = (C(A_i)\times K_j) \cup (A_i\times C(K_j)) \to E'_{i+j-1}$ defined as follows:

$$[\chi(\tilde{\lambda}_i), \chi(\rho_j)]|_{C(A_i)\times K_j} : C(A_i)\times K_j \xrightarrow{\chi(\tilde{\lambda}_i)\times 1} E^{(d\cdot i+s)}|_{B_i}\times F_{j-1} \\ \subseteq E^{(d\cdot(i+1)+s-1)}|_{B_i}\times F_{j-1} \xrightarrow{\psi_{i,j-1}} E'_{i+j-1}|_{B_i} \subseteq E'_{i+j-1},$$

$$[\chi(\tilde{\lambda}_i), \chi(\rho_j)]|_{A_i\times C(K_j)} : A_i\times C(K_j) \xrightarrow{\tilde{\lambda}_i\times\chi(\rho_j)} E^{(d\cdot i+s-1)}|_{B_{i-1}}\times F_j \\ \xrightarrow{\psi_{i-1,j}} E'_{i+j-1}|_{B_{i-1}} \subseteq E'_{i+j-1}.$$

This implies immediately that $\mathrm{Cat}\,(E'_k) \leq \mathrm{Cat}\,(E'_{k-1}) + 1$. Then by induction on k, we obtain that $\mathrm{Cat}\,(E'_k) \leq k$. Thus we have $\mathrm{Cat}\,(X) = \mathrm{Cat}\,(E\times_G F) = \mathrm{Cat}\,(E'_{m+n}) \leq m+n \leq m+\frac{\dim B}{d}$. This completes the proof of Theorem 2.3.

4 Proof of Theorem 2.9

As is well known, we have the following principal bundles (see for example [2], [34] and [9] in particular for the last fibration):

$$\begin{aligned} &\mathrm{Sp}(1) \longrightarrow \mathrm{Sp}(2) \longrightarrow S^7, \\ &\mathrm{SU}(3) \longrightarrow \mathrm{SU}(4) \longrightarrow S^7, \\ &G_2 \longrightarrow \mathrm{Spin}(7) \longrightarrow S^7, \\ &\mathrm{Spin}(7) \longrightarrow \mathrm{Spin}(9) \longrightarrow S^{15}, \\ &G_2 \longrightarrow \mathrm{Spin}(8) \longrightarrow S^7 \times S^7. \end{aligned}$$

Each scalar matrix $(-1) \in \mathrm{Sp}(2)$ and $(-1) \in \mathrm{SU}(4)$ acts on S^7 as the antipodal map, and so does the center of Spin(7). Similarly the center of Spin(9) acts on S^{15} as the antipodal map. Recall that the center of Spin(8) is isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$, each generator of which acts on S^7 as the antipodal map respectively. Since there are isomorphisms $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ and $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$, we obtain principal bundles:

$$\begin{aligned} &\mathrm{Sp}(1) \longrightarrow \mathrm{SO}(5) \longrightarrow \mathbb{R}P^7, \\ &\mathrm{SU}(3) \longrightarrow \mathrm{SO}(6) \longrightarrow \mathbb{R}P^7, \\ &\mathrm{G}_2 \longrightarrow \mathrm{SO}(7) \longrightarrow \mathbb{R}P^7, \\ &\mathrm{Spin}(7) \longrightarrow \mathrm{SO}(9) \longrightarrow \mathbb{R}P^{15}, \\ &\mathrm{G}_2 \longrightarrow \mathrm{PO}(8) \longrightarrow \mathbb{R}P^7 \times \mathbb{R}P^7. \end{aligned}$$

Cone decompositions of the fibres except Spin(7) are given as follows (see Theorem 2.1 of [13] for G_2):

$$
\begin{aligned}
&* \subset \mathrm{Sp}(1) = S^3,\\
&* \subset \mathrm{SU}(3)^{(5)} \subset \mathrm{SU}(3),\\
&* \subset \mathrm{G}_2^{(5)} \subset \mathrm{G}_2^{(8)} \subset \mathrm{G}_2^{(11)} \subset \mathrm{G}_2,
\end{aligned}
$$

where $\mathrm{SU}(3)^{(5)} = \mathrm{G}_2^{(5)} = \Sigma\mathbb{C}P^2$, $\mathrm{SU}(3) = \mathrm{SU}(3)^{(5)} \cup CS^7$, $\mathrm{G}_2^{(8)} \simeq \mathrm{G}_2^{(5)} \cup C(S^5 \cup e^7)$, $\mathrm{G}_2^{(11)} \simeq \mathrm{G}_2^{(8)} \cup C(S^8 \cup e^{10})$ and $\mathrm{G}_2 = \mathrm{G}_2^{(11)} \cup CS^{13}$. Since these fibres satisfy the conditions in Remark 2.2 (1), we obtain $\mathrm{Cat}\,(\mathrm{SO}(5)) \leq 8$, $\mathrm{Cat}\,(\mathrm{SO}(6)) \leq 9$, $\mathrm{Cat}\,(\mathrm{SO}(7)) \leq 11$ and $\mathrm{Cat}\,(\mathrm{PO}(8)) \leq 18$ using Theorem 2.3. By virtue of the mod 2 cup-lengths we have that $\mathrm{cup}(\mathrm{SO}(5)) \geq 8$, $\mathrm{cup}(\mathrm{SO}(6)) \geq 9$, $\mathrm{cup}(\mathrm{SO}(7)) \geq 11$ and $\mathrm{cup}(\mathrm{PO}(8)) \geq 18$ respectively. Thus we obtain the results for SO(5), SO(6), SO(7) and PO(8).

A cone decomposition of Spin(7) is given as follows in [14]:

$$* = F_0 \subset F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 = \mathrm{Spin}(7),$$

where $F_1 = \mathrm{SU}(4)^{(7)}$, $F_2 = \mathrm{SU}(4)^{(12)} \cup e^6$, $F_3 = \mathrm{SU}(4) \cup e^6 \cup e^9 \cup e^{11} \cup e^{13}$ and $F_4 = \mathrm{Spin}(7)^{(18)}$. We need here to check if the filtration satisfies Assumption 1; the only problem is to determine whether $\psi|_{\mathrm{Spin}(7)^{(3)} \times F_1} : \mathrm{Spin}(7)^{(3)} \times F_1 \to F$ is compressible into F_4 or not. Since $\mathrm{Spin}(7)^{(3)}$ and F_1 are included in $\mathrm{SU}(4) \subset F_4$, we have $\mathrm{Im}\,(\psi|_{\mathrm{Spin}(7)^{(3)} \times F_1}) \subset F_4$. Then we obtain $\mathrm{Cat}\,(\mathrm{SO}(9)) \leq 20$ using Theorem 2.3. The mod 2 cup-length implies that $\mathrm{cup}(\mathrm{SO}(9)) \geq 20$. Thus we obtain the result for SO(9).

Since SO(8) is homeomorphic to $\mathrm{SO}(7) \times S^7$, we easily see that

$$\mathrm{Cat}\,(\mathrm{SO}(8)) \leq \mathrm{Cat}\,(\mathrm{SO}(7)) + \mathrm{Cat}\,(S^7) = 12$$

by Takens [**?**]. The mod 2 cup-length implies that $\mathrm{cup}(\mathrm{SO}(8)) \geq 12$. Thus we obtain the result for SO(8). This completes the proof of Theorem 2.9.

5 Proof of Theorems 2.7 and 2.8

Firstly, we show Theorem 2.7. The following principal bundle is well-known:

$$\mathrm{SU}(n-1) \longrightarrow \mathrm{SU}(n) \longrightarrow S^{2n-1}.$$

The central (cyclic) subgroup C_m of $\mathrm{SU}(n)$ acts on S^{2n-1} freely and hence we obtain a principal bundle:

$$\mathrm{SU}(n-1) \longrightarrow \mathrm{SU}(n)/C_m \longrightarrow L^{2n-1}(m),$$

where $L^{2n-1}(m)$ is a lens space of dimension $2n-1$.

A cone decomposition of $\mathrm{SU}(n-1)$ is constructed by Kadzisa [17]:

$$* \subset V \subset V^2 \subset \cdots \subset V^{n-2} = SU(n-1),$$

where $V^k \subseteq SU(n-1)$ is a representing subspace of the quotient module $H^*(SU(n-1))/D^{k+1}$ and D^{k+1} is the submodule generated by products of $k+1$ elements in positive degrees, which satisfies $V^i \cdot V^j \subseteq V^{i+j}$ for any i and j. Thus V is the subcomplex $S^3 \cup e^5 \cup e^7 \cup \cdots \cup e^{2n-3}$ of $\mathrm{SU}(n-1)$ which is homeomorphic to $\Sigma\mathbb{C}P^{n-2}$ (see [33], for example). Then Assumption 1 is automatically satisfied, and hence using $\mathrm{SU}(n-1)^{(k)} \subset V^k$, we obtain

$$\mathrm{Cat}\,(\mathrm{SU}(n)/C_m) \leq 3(n-1)$$

by Theorem 2.3. This completes the proof of Theorem 2.7.

Secondly, we show Theorem 2.8. By Rudyak [20] [21] and Strom [30], we know the following proposition.

Proposition 5.1 (Rudyak [20] [21], Strom [30]) *Let h be a cohomology theory. For an element $u \in \tilde{h}^*(X)$, let $\mathrm{wgt}(u;h)$ be the minimal number k such that $(e^X_k)^*(u) \neq 0$ where $e^X_k : P^k\Omega X \to P^\infty \Omega X \simeq X$, which satisfies*

(1) We have $\mathrm{wgt}(0;h) = \infty$ and $\infty > \mathrm{wgt}(u;h) \geq 1$ for any $u \neq 0$ in $\tilde{h}^*(X)$.
(2) For any cohomology theory h, we have

$$\min\{\mathrm{wgt}(u;h), \mathrm{wgt}(v;h)\} \leq \mathrm{wgt}(u+v;h).$$

(3) For any multiplicative cohomology theory h, we have

$$\mathrm{wgt}(u;h) + \mathrm{wgt}(v;h) \leq \mathrm{wgt}(u \cdot v;h).$$

(4) $\mathrm{wgt}(X;h) = \max\{\mathrm{wgt}(u;h) \mid u \in \tilde{h}^*(X), u \neq 0\}$.

Le us recall that, for any compact Lie group G, the ordinary cohomology of ΩG is concentrated in even degrees. Then, for any element u of even degree in $\tilde{H}^*(G;\mathbb{Z}/p)$, we have $\mathrm{wgt}(u;H\mathbb{Z}/p) \geq 2$, since $P^1(\Omega G) = \Sigma\Omega(G)$.

The cohomology rings of $\mathrm{SU}(p^r)/C_{p^s}$ for a prime p and $1 \leq s \leq r$ are given as follows (see [3]):

$$H^*(\mathrm{SU}(p^r)/C_{p^s};\mathbb{Z}/p) = \mathbb{Z}/p[x_2]/(x_2^{p^r}) \otimes \wedge(x_1, x_3, \ldots, x_{2p^r-3}).$$

Note that $x_1^2 = x_2$ if $p = 2$ and $s = 1$. Then, using Proposition 5.1, we obtain

$$\mathrm{wgt}(\mathrm{SU}(p^r)/C_{p^s};H\mathbb{Z}/p) \geq \mathrm{wgt}(x_1 \cdot x_2^{p^r-1} \cdot x_3 \cdot \cdots \cdot x_{2p^r-3};H\mathbb{Z}/p) \geq 3(p^r-1),$$

since $\mathrm{wgt}(x_2;H\mathbb{Z}/p) \geq 2$. Thus we have the following lemma.

Lemma 5.2 *$r\mathrm{cat}\,(\mathrm{SU}(p^r)/C_{p^s}) \geq 3(p^r-1)$ for any prime p and $1 \leq s \leq r$.*

By using Theorem 2.7, we obtain Theorem 2.8.

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