CATEGORICAL LENGTH, RELATIVE L-S CATEGORY AND HIGHER HOPF INVARIANTS

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Abstract. We first introduce a homotopy-theoretical version of Fox's categorical sequence in terms of a new reltive L-S cateory, which gives a better upper estimate 'the categorical length' for the L-S category than Ganea's cone length. Then we discuss how higher Hopf invariants fit with the categorical sequence through our relative L-S category. We also clarify the relations among our new relative L-S category and other three known relative L-S categories introduced by Fadell and Husseini, by Berstein and Ganea and by Arkowitz and Lupton. The main goal of this paper is to show that the categorical length is equal to the L-S category. In addition, the definition of cup length and module weight for our relative L-S category are given.

1. Introduction

Throughout this paper, we work in $\mathcal{T}$ the category of topological spaces and maps. A closed subset is always assumed to be a neighbourhood deformation retract, and a pair is assumed to be an NDR-pair in the sense of G. Whitehead [21]. The one-point-space is denoted by $*$. The (normalised) Lusternik-Schnirelmann category $\mathrm{cat}(X)$, L-S category for short, is introduced in [16] as the least number m such that there is a covering of X by $m{+}1$ closed subsets U_j, $0 \leq j \leq m$, where each U_j is contractible in X. By modifying the idea due to R. Fox [7], T. Ganea [8] gives the following definition of a strong version of L-S category for a space X: the strong L-S category $\mathrm{Cat}(X)$ is the least number m such that there is a space $Y \simeq X$ with a covering of Y by $m{+}1$ closed subsets U_j, $0 \leq j \leq m$ where each U_j is contractible in itself. By Ganea [8], it is shown that

$$\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1.$$

Remark 1.1. *Fadell and Husseini* [6] *introduced a notion of a relative L-S category as follows: for a pair (K, A), $\mathrm{cat}^{\mathrm{FH}}(K, A)$ is given as the least number m such that there is a covering of K by $m{+}1$ closed subsets $V \supset A$ and U_j, $1 \leq j \leq m$ where V is compressible relative A into A in K and each U_j is contractible in K. It is also clear by definition that $\mathrm{cat}^{\mathrm{FH}}(K, *) = \mathrm{cat}(K)$.*

These definitions, however, do not suggest any effective way to compute the (strong) L-S category but do suggest how to give some upper estimates: in [7], Fox introduced a notion of 'categorical sequence' for a space X as a sequence

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$F_0 \subset \cdots \subset F_i \subset \cdots \subset F_m$ of closed subsets such that $F_0 \simeq *$ in X, $F_m = X$ and $F_i \smallsetminus F_{i-1}$ is contractible in X, $i > 0$. It is also shown by Fox that the least such number m gives exactly the L-S category of X. But unfortunately, we did not know any effective way to construct a categorical sequence.

Similar to the categorical sequence, Ganea introduced in [8] a notion of 'cone decomposition' for a space X as a sequence $F_0 \subset \cdots \subset F_i \subset \cdots \subset F_m$ of closed subsets such that $F_0 \simeq *$, $F_m = X$ and $F_i \simeq F_{i-1} \cup_{h_i} C(K_i)$, $i > 0$. It is also shown by Ganea that the least such number m gives exactly the strong L-S category of X. Unlike the categorical sequence, we can construct a cone decomposition using a cell-decomposition of a space, if one knows an explicit definition of the given space. Thus the cone decomposition gives a nice upper estimate if the given space is not too complicated. For a complicated space X, we could not know whether $\mathrm{cat}(X) = \mathrm{Cat}(X)$ or $\mathrm{cat}(X) = \mathrm{Cat}(X) - 1$.

By G. Whitehead [21], the definition of L-S category is interpreted in terms of deformation of a diagonal map as the following definition for a space X.

Definition 1.2. *The L-S category* $\mathrm{cat}(X)$ *of* X *is the least number* m *such that the* $m{+}1$ *fold diagonal map* $\Delta^{m+1} : X \to \prod^{m+1} X$ *is compressible into* $\mathrm{T}^{m+1} X = \{(x_0, x_1, ..., x_m) \in \prod^{m+1} X \mid \exists i\, x_i = *\} \subseteq \prod^{m+1} X$ *the 'fat wedge'.*

Similarly to the above, one can give an alternative definition of a relative L-S category for a pair (K, A) to fit with Whitehead's definition of L-S category.

Definition 1.3. *Let* $A \subseteq K$*. Then the L-S category* $\mathrm{cat}(K, A)$ *is the least number* $m \geq 0$ *such that restriction to* K *of the* $m{+}1$ *fold diagonal map* $\Delta_K^{m+1} : K \to \prod^{m+1} K$ *is compressible relative* A *into* $\mathrm{T}^{m+1}(K, A) = A \times \prod^m K \cup K \times \mathrm{T}^m K \subseteq \prod^{m+1} K$ *the 'fat wedge' of a pair* (K, A)*.*

Remark 1.4. *For any map* $f : A \to K$*, we may assume that* f *is an inclusion up to homotopy, and hence the definition of relative L-S category implies a definition of* $\mathrm{cat}^{\mathrm{FH}}(f)$ *the L-S category of* f *in the sense of Fadell and Husseini.*

In the present paper, we alter the Fox's definition of a categorical sequence to fit with Whitehead's definition of L-S category:

Definition 1.5. *A categorical sequence for a space* X *is a sequence of closed subspaces* $F_0 \subset \cdots \subset F_i \subset \cdots \subset F_m$ *such that* $F_m \simeq X$, $F_0 \simeq *$ *in* X *and* $\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m$ *is compressible into* $F_{i-1} \times F_m \cup F_m \times *$ *relative* F_{i-1} *for any* $i > 0$*, where we identify* F_{i-1} *with its diagonal image in* $F_{i-1} \times F_{i-1} \subset F_{i-1} \times F_m \cup F_m \times *$*. Let us call the least such* $m \geq 0$ *the 'categorical length' of* X *and denote by* $\mathrm{catlen}(X)$*.*

Inspired by the definition of a relative L-S category due to Fadell and Husseini, we introduce a relative version of categorical sequence as follows:

Definition 1.6. *A categorical sequence for a pair* (X, A) *is a sequence of pairs* $(F_0, A) \subset \cdots \subset (F_i, A) \subset \cdots \subset (F_m, A)$ *such that* $(F_m, A) \simeq (X, A)$ *relative* A*,* $F_0 \simeq A$ *relative* A *in* X *and* $\Delta_i : F_i \xrightarrow{\Delta} F_i \times F_i \subset F_m \times F_m$ *is compressible into* $F_{i-1} \times F_m \cup F_m \times A$ *relative* F_{i-1}*,* $i > 0$*. Let us call the least such* $m \geq 0$ *the 'categorical length' of* (X, A) *and denote by* $\mathrm{catlen}(X, A)$*.*

To describe the categorical sequence in terms of a relative L-S category, we give a definition of a new extended version of relative L-S category: from now on, we work in the category $\mathcal{T}^A$, in which an object is a pair (X, A) with an inclusion $i^X : A \hookrightarrow X$ and a morphism is a map of pairs $f : (X, A) \to (Y, A)$ with $i^Y = f \circ i^X$. We remark that, if $A = *$ the one point space, then $\mathcal{T}^A$ is the usual category of connected spaces and based maps. We say that $(X, K; A)$ is a pair in $\mathcal{T}^A$ when (X, A) and (K, A) are objects in $\mathcal{T}^A$ and (X, K) is a pair in $\mathcal{T}$, that $(X, K, L; A)$ is a triple in $\mathcal{T}^A$ when (X, A), (K, A), (L, A) are objects in $\mathcal{T}^A$ and (X, K, L) is a triple in $\mathcal{T}$, and that $(X; K, L; A)$ is a triad in $\mathcal{T}^A$ when (X, A), (K, A), (L, A) are objects in $\mathcal{T}^A$ and $(X; K, L)$ is a triad in $\mathcal{T}$.

We remark, for any pair $(X, K; A)$ in $\mathcal{T}^A$, that the diagonal image of A in $\prod^{m+1} X$ is in the subspace $\mathrm{T}^{m+1}(X, L)$. Thus for any $(X, A) \supset (L, A) \in \mathcal{T}^A$, we regard $(\prod^{m+1} X, A) \supset (\mathrm{T}^{m+1}(X, L), A) \in \mathcal{T}^A$.

Definition 1.7. *Let $(X; K, L; A)$ be a triad in $\mathcal{T}^A$. Then $\mathrm{cat}(X; K, L; A)$ is the least number m such that the restriction of the m+1 fold diagonal map of X to K, $\Delta^{m+1}|_K : K \to \prod^{m+1} X$, is compressible relative A into $\mathrm{T}^{m+1}(X, L)$.*

Definition 1.8. *Let $(X; K, L; A)$ be a triad in $\mathcal{T}^A$. Then $\mathrm{Cat}(X; K, L; A)$ is the least number m such that there is a space $Y \simeq X$ relative A with a covering of Y by m+1 closed subsets $V \supset A$ and U_j, $1 \leq j \leq m$ where A is a deformation retract of V and each U_j is contractible in itself.*

Using Harper's arguments on the homotopy of maps to the total space of a fibration in [10], Cornea [4] has given a proof of the following:

Proposition 1.9. *Let (X, A) be an object in $\mathcal{T}^A$, $(Y, K; A)$ be a pair in $\mathcal{T}^A$ with the inclusion $j : (K, A) \hookrightarrow (Y, A)$ and $f : (X, A) \to (Y, A)$ be a map in $\mathcal{T}^A$. If $f|_X : X \to Y$ has a compression $\sigma : X \to K$ such that $j \circ \sigma \sim f$ and $\sigma \circ i^X \sim i^K$ in $\mathcal{T}$, then there is a map $\sigma' : (X, A) \to (K, A)$ a compression relative A of f such that $\sigma \sim \sigma'|_X : X \to K$.*

One of its direct consequence is described as follows.

Corollary 1.10. *Let $(X; K.L; A)$ be a triple in $\mathcal{T}^A$. Then $\mathrm{cat}(X; K, L; A)$ is the same as the least number m such that $\Delta^{m+1}|_K : K \to \prod^{m+1} X$ is compressible to a map $s : K \to \mathrm{T}^{m+1}(X, L)$ such that $s|_A$ is homotopic to the diagonal map $\Delta_A : A \to \prod^{m+1} A \subset \mathrm{T}^{m+1}(X, L)$.*

Remark 1.11.
1. $\mathrm{cat}(X; X, *; *) = \mathrm{cat}(X)$ *and* $\mathrm{cat}(X; *, *; *) = 0$.
2. *We often abbreviate* $\mathrm{cat}(X; X, L; A)$ *by* $\mathrm{cat}(X, L; A)$, $\mathrm{cat}(X; K, A; A)$ *by* $\mathrm{cat}(X; K, A)$, $\mathrm{cat}(X; X, A)$ *by* $\mathrm{cat}(X, A)$ *and* $\mathrm{cat}(X; K, *)$ *by* $\mathrm{cat}(X; K)$.
3. *We may replace inclusions $(L, A) \hookrightarrow (X, A)$ and $(K, A) \hookrightarrow (X, A)$ by maps $f : (L, A) \to (X, A)$ and $g : (K, A) \to (X, A)$ in $\mathcal{T}^A$, since every such map is an inclusion map up to homotopy relative A by taking the mapping cylinder of $K \cup_A L \xrightarrow{g \cup_A f} X$. Then we often denote* $\mathrm{cat}(X; K, L; A)$ *by* $\mathrm{cat}(g, f)$. *By applying (1), we have* $\mathrm{cat}(g, *) = \mathrm{cat}(g)$.

Note that there are two other relative L-S categories by Berstein and Ganea [2] and by Arkowitz and Lupton [1].

Remark 1.12. *Arkowitz and Lupton defined their relative L-S category for a map $h : X \to Y$. Since a map is up to homotopy a fibration, we may assume that h is a fibration with fibre $L = h^{-1}(*) \subset X$. Then the relative L-S category of h in the sense of Arkowitz and Lupton is depending only on the pair (X, L) by its definition. Thus we often denote it by* $\mathrm{cat}^{\mathrm{AL}}(X, L)$ *in this paper.*

In §3, we show the following relationship of our extended version of relative L-S category with existing the three known relative L-S categories $\mathrm{cat}^{\mathrm{FH}}(X, A)$ by Fadell and Husseini, $\mathrm{cat}^{\mathrm{BG}}(X, K)$ by Berstein and Ganea and $\mathrm{cat}^{\mathrm{AL}}(X, L)$ (see Remark 1.12 above) by Arkowitz and Lupton.

Theorem 3.1. *The known three relative L-S categories are described to be special cases of our new relative L-S category as follows:*

(1) *Let* $X \supset K \supset L = A = *$*. Then* $\mathrm{cat}(X; K, *; *) = \mathrm{cat}^{\mathrm{BG}}(X, K)$ *the realtive L-S category in the sense of Berstein and Ganea* [2]*. More generally for a map* $g : K \to X$ *in* $\mathcal{T}_*$*, we have* $\mathrm{cat}(g, *) = \mathrm{cat}^{\mathrm{BG}}(g)$.

(2) *Let* $X = K \supset L = A \supset *$*. Then* $\mathrm{cat}(X; X, A; A) = \mathrm{cat}^{\mathrm{FH}}(X, A)$ *the relative L-S category in the sense of Fadell and Husseini* [6].

(3) *Let* $h : X \to Y$ *be a fibration with fibre* $L \subset X$ *and* $K = X \supset L \supset A = *$*. Then* $\mathrm{cat}(X; X, L; *) = \mathrm{cat}^{\mathrm{AL}}(X, L)$ *the relative L-S category in the sense of Arkowitz and Lupton* [1].

We also introduce a new higher Hopf invariant: let $(X; K, L; A)$ be a triad in $\mathcal{T}^A$, V be a co-loop co-H-space, i.e, a one-point-union of a 1-connected co-H-space with finitely-many circles, and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. If $\mathrm{cat}(X; K, L; A) \leq m$, then a relative higher Hopf invariant $H_m^{(X;K,L;A)}(\alpha)$ is defined as a subset of $[V, \Omega(X, L) * \Omega(X) * \underset{m}{\cdots} * \Omega(X)]$. If $K \supset L$ and $\mathrm{cat}(K; K, L; A) \leq m$, then an absolute higher Hopf invariant $H_m^{(K,L;A)}(\alpha)$ is defined as a subset of $[V, \Omega(K, L) * \Omega(K) * \underset{m}{\cdots} * \Omega(K)]$ (see §4 for more details). The following result clarifies how a higher Hopf invariant determines whether a cone decomposition reduces to a categorical sequence or not.

Theorem 4.3. *Let* $(X; K, L; A)$ *be a triad in* $\mathcal{T}^A$*,* V *be a co-loop co-H-space and* $\alpha : V \to K$ *be a map in* $\mathcal{T}$ *such that* $X \supset \hat{K} = K \cup_\alpha CV \supset K$*. If* $\mathrm{cat}(X; K, L; A) \leq m$ *and* $H_m^{(X;K,L;A)}(\alpha) = 0$*, then* $\mathrm{cat}(X; \hat{K}, L; A) \leq m$.

We often abbreviate $H_m^{(X;K,A;A)}(\alpha)$ by $H_m^{(X;K,A)}(\alpha)$, $H_m^{(X;K,*)}(\alpha)$ by $H_m^{(X;K)}(\alpha)$, $H_m^{(K,A;A)}(\alpha)$ by $H_m^{(K,A)}(\alpha)$ and $H_m^{(K,*)}(\alpha)$ by $H_m^{K}(\alpha)$. Note that the definition of the absolute higher Hopf invariant $H_m^{K}(\alpha)$ coincides with the ordinary definition of the higher Hopf invariant $H_m(\alpha)$ in the sense of [12].

The main goal of this paper is stated as follows:

Theorem 5.2. *For any* X *in* $\mathcal{T}$*, we have* $\mathrm{cat}(X) = \mathrm{catlen}(X)$*. More generally, for any object* $(X, A) \in \mathcal{T}^A$*, we have* $\mathrm{cat}^{\mathrm{FH}}(X, A) = \mathrm{catlen}(X, A)$.

Corollary 5.4. *Let* (X, A) *be an object in* $\mathcal{T}^A$*. If* $\mathrm{cat}^{\mathrm{FH}}(X, A) = m > 0$*, then there exists a sequence for pairs* $\{(F_i, A)\,;\, 0{\leq}i{\leq}m\}$ *such that* $(F_0, A) \simeq (A, A)$ *in* (F_m, A)*,* $(F_m, A) \simeq (X, A)$ *relative* A *and* $\mathrm{cat}(X; F_i, A) \leq i$*,* $i > 0$*. Moreover we have* $\mathrm{cat}(F_m/F_{i-1}; F_i/F_{i-1}) \leq 1$ *with a partial co-action* $F_i \to F_m/F_{i-1} \vee F_m$ *along the collapsion* $F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$*,* $i > 0$*. In particular,* F_m/F_{m-1} *is a co-H-space coacting on* F_m *along the collapsion* $F_m \to F_m/F_{m-1}$.

2. A_∞-decomposition of a map

In [8], Ganea introduced a so-called 'fibre-cofibre' construction for a map, which can be interpreted as the pullback construction from the view-point of Definition 1.3 the definition of relative L-S category by Fadell and Husseini [6]. We may regard this construction as an A_∞-decomposition of a map using the pushout-pullback diagram (see [11, Lemma 2.1] and also Sakai [18] for the detailed proof in a general context): let $(X;K,L;A)$ be a triad in $\mathcal{T}^A$.

Let us recall that, in $\mathcal{T}$, the homotopy fibre of $\mathrm{T}^m_{i=0}(X,A_i) \hookrightarrow \prod^{m+1} X$ has the homotopy type of the join $\Omega(X,A_0) * \underset{m+1}{\cdots} * \Omega(X,A_m)$ by Ganea. We denote by $E^m(\Omega(X)) = \Omega(X) * \underset{m}{\cdots} * \Omega(X)$ which has the homotopy type of the homotopy fibre of $\mathrm{T}^m(X,*) \hookrightarrow \prod^m X$. The homotopy fibre of the inclusion $\mathrm{T}^{m+1}(X,L) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $E^{m+1}(\Omega(X,L)) = \Omega(X,L) * \Omega(X) * \underset{m}{\cdots} * \Omega(X)$: by the homotopy pushout-pullback diagram in $\mathcal{T}$, which is given by [11, Lemma 2.1] with $(Y,B) = (\prod^m X, \mathrm{T}^m X)$, $Z = *$ and $f = g = *$.

$$
\begin{tikzcd}
\Omega(X,L)\times E^m(\Omega(X)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\mathrm{HPO}"] & E^m(\Omega(X)) \arrow[d] & \\
\Omega(X,L) \arrow[r] & E^{m+1}(\Omega(X,L)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{HPB}"] & \mathrm{T}^{m+1}(X,L) \arrow[d] \\
& * \arrow[r] & \prod^{m+1} X.
\end{tikzcd}
\tag{2.1}
$$

Thus we see that the homotopy fibre of the inclusion $\mathrm{T}^{m+1}(X,L) \hookrightarrow \prod^{m+1} X$ has the homotopy type of $\Omega(X,L) * E^m(\Omega(X)) = E^{m+1}(\Omega(X,L))$ by the induction hypothesis.

Similarly, we define $P^m(\Omega(X,L))$ inductively from $P^0(\Omega(X,L)) = L$ as the homotopy pushout in the following homotopy pushout-pullback diagram which is given by [11, Lemma 2.1] with $(Y,B) = (\prod^m X, \mathrm{T}^m X)$, $Z = X$ and $(f,g) = (1_X, \Delta^m_X)$:

$$
\begin{tikzcd}
E^m(\Omega(X,L)) \arrow[r, "p^{\Omega(X,L)}_{m-1}"] \arrow[d] \arrow[dr, phantom, "\mathrm{HPO}"] & P^{m-1}(\Omega(X,L)) \arrow[d] & \\
* \arrow[r] & P^m(\Omega(X,L)) \arrow[r, "q^{(X,L)}_m"] \arrow[d, "e^{(X,L)}_m"'] \arrow[dr, phantom, "\mathrm{HPB}"] & \mathrm{T}^{m+1}(X,L) \arrow[d] \\
& X \arrow[r, "\Delta^{m+1}"] & \prod^{m+1} X,
\end{tikzcd}
\tag{2.2}
$$

where $q^{(X,L)}_m$ covers the diagonal map $\Delta^{m+1} : X \to \prod^{m+1} X$. Then we define $p^{\Omega(X,L)}_{m+1} : E^{m+1}(\Omega(X,L)) \to P^m(\Omega(X,L))$ as the homotopy fibre of $e^{(X,L)}_m : P^m(\Omega(X,L)) \to X$ given in the diagram, where $e^{(X,L)}_0 : L \hookrightarrow X$ is nothing but the canonical inclusion. These constructions due to Ganea [8] yields the following ladder of fibrations which have the same fibre $\Omega(X)$, giving a generalisation of an

A_∞-structure (see Stasheff [19]):

$$
\begin{tikzcd}
\Omega(X,L) \arrow[r, hook, "\overset{*}{\wr}"] \arrow[d, "p_1^{\Omega(X,L)}"] & \cdots \arrow[r, hook, "\overset{*}{\wr}"] & E^{m+1}(\Omega(X,L)) \arrow[r, hook, "\overset{*}{\wr}"] \arrow[d, "p_{m+1}^{\Omega(X,L)}"] & \cdots \arrow[r, hook, "\overset{*}{\wr}"] & E^{\infty}(\Omega(X,L)) \arrow[d, "p_{\infty}^{\Omega(X,L)}"] \\
L \arrow[r, hook] & \cdots \arrow[r, hook] & P^{m}(\Omega(X,L)) \arrow[r, hook] & \cdots \arrow[r, hook] & P^{\infty}(\Omega(X,L))
\end{tikzcd}
\tag{2.3}
$$

with $e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \underset{m}{\cup} P^m(\Omega(X,L)) \to X$ given by $e_\infty^{(X,L)}|_{P^m(\Omega(X,L))} = e_m^{(X,L)}$ with fibre $E^\infty(\Omega(X,L))$. Since $E^\infty(\Omega(X,L)) = \underset{m}{\cup} E^m(\Omega(X,L))$ is weekly contractible, $e_\infty^{(X,L)} : P^\infty(\Omega(X,L)) = \underset{m}{\cup} P^m(\Omega(X,L)) \to X$ is a weekly equivalence. If further X is a CW complex, then there is a right homotopy inverse $h^{(X,L)} : X \to P^\infty(\Omega(X,L))$ of $e_\infty^{(X,L)}$, where $h^{(X,L)}$ is also a weak equivalence.

The ladder (2.3) is natural with respect to a map of triads in $\mathcal{T}^A$ as follows.

Lemma 2.1. *For any map $f : (X;K,L;A) \to (X';K',L';A)$ of triads in $\mathcal{T}^A$, there is the following commutative diagram with $f|_{(X,L)} : (X,L) \to (X',L')$ and $f|_L : L \to L'$ the restrictions of f.*

$$
\begin{tikzcd}
E^m(\Omega(X,L)) \arrow[rr, hook] \arrow[dr, "E^m(\Omega(f|_{(X,L)}))"] \arrow[dd, "p_1^{\Omega(X,L)}"'] & & E^{m+1}(\Omega(X,L)) \arrow[dr, "E^{m+1}(\Omega(f|_{(X,L)}))"] \arrow[dd, "p_{m+1}^{\Omega(X,L)}" near start] & \\
& E^m(\Omega(X',L')) \arrow[rr, hook] \arrow[dd, "p_m^{\Omega(X',L')}" near start] & & E^{m+1}(\Omega(X',L')) \arrow[dd, "p_{m+1}^{\Omega(X',L')}"] \\
P^{m-1}(\Omega(X,L)) \arrow[rr, hook] \arrow[dr, "P^{m-1}(\Omega(f|_{(X,L)}))"'] & & P^m(\Omega(X,L)) \arrow[dr] & \\
& P^{m-1}(\Omega(X',L')) \arrow[rr, hook, "P^m(\Omega(f|_{(X,L)}))"] & & P^m(\Omega(X',L'))
\end{tikzcd}
$$

We give here another kind of naturality of the ladder (2.3) in $\mathcal{T}^A$ induced from the structure map $\sigma : K \to P^m(\Omega(X,L))$ of $\mathrm{cat}(X;K,L;A) \leq m$.

Lemma 2.2. *For any triad $(X;K,L;A)$ in $\mathcal{T}^A$ with a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$, there is a sequence of maps $\sigma_n : P^n(\Omega(X,K)) \to P^{m+n}(\Omega(X,L))$ $(n \geq 0)$ with $\sigma_0 = \sigma$, which makes the following diagram commutative up to homotopy relative A.*

$$
\begin{tikzcd}
P^{n-1}(\Omega(X,K)) \arrow[r, hook] \arrow[d, "\sigma_{n-1}"] & P^n(\Omega(X,K)) \arrow[r, "e_n^{(X,K)}"] \arrow[d, "\sigma_n"] & X \arrow[d, "\mathrm{id}_X"'] \\
P^{m+n-1}(\Omega(X,L)) \arrow[r, hook] & P^{m+n}(\Omega(X,L)) \arrow[r, "e_{m+n}^{(X,L)}"] & X
\end{tikzcd}
\tag{2.4}
$$

Proof: We construct σ_n inductively on $n \geq 1$: assuming that we have done up to $n-1$, we consider σ_n. The homotopy commutativity relative A of the (2.5) without the dotted arrow induces a map of fibres in $\mathcal{T}$, namely $\hat{\sigma}_n : E^n(\Omega(X,K)) \to$

$E^{m+n}(\Omega(X,L))$.

$$
\begin{tikzcd}
E^n(\Omega(X,K)) \arrow[r, "p_n^{\Omega(X,K)}"] \arrow[d, dotted, "\hat{\sigma}_n"] & P^{n-1}(\Omega(X,L)) \arrow[r, "e_{n-1}^{(X,K)}"] \arrow[d, "\sigma_{n-1}"] & X \arrow[d, "\mathrm{id}_X"] \\
E^{m+n}(\Omega(X,L)) \arrow[r, "p_{m+n}^{\Omega(X,L)}"] & P^{m+n-1}(\Omega(X,L)) \arrow[r, "e_{m+n-1}^{(X,L)}"] & X.
\end{tikzcd} \tag{2.5}
$$

Using a standard argument in the homotopy theory, the homotopy commutativity of the left-hand square of the diagram (2.5) with dotted arrow $\hat{\sigma}$ implies the existence of $\sigma_n : P^n(\Omega(X,L)) \to P^{m+n}(\Omega(X,L)$ which makes the diagram (2.4) commutative up to homotopy relative A.

Thus there is a sequence of maps σ_n $(n \geq 0)$ and $\hat{\sigma}_n$ $(n \geq 1)$ which makes the diagram (2.4) commutative up to homotopy. □ □

3. Properties of a new relative L-S category

Our new relative L-S category enjoys the following relationship with the known three different relative L-S categories:

Theorem 3.1. *The known three relative L-S categories are described to be special cases of our new relative L-S category as follows:*

(1) *Let $X \supset K \supset L = A = *$. Then $\mathrm{cat}(X; K, *; *) = \mathrm{cat}^{\mathrm{BG}}(X, K)$ the realtive L-S category in the sense of Berstein and Ganea* [2]. *More generally for a map $g : K \to X$ in $\mathcal{T}_*$, we have $\mathrm{cat}(g, *) = \mathrm{cat}^{\mathrm{BG}}(g)$.*
(2) *Let $X = K \supset L = A \supset *$. Then $\mathrm{cat}(X; X, A; A) = \mathrm{cat}^{\mathrm{FH}}(X, A)$ the relative L-S category in the sense of Fadell and Husseini* [6].
(3) *Let $h : X \to Y$ be a fibration with fibre $L \subset X$ and $K = X \supset L \supset A = *$. Then $\mathrm{cat}(X; X, L; *) = \mathrm{cat}^{\mathrm{AL}}(X, L)$ the relative L-S category in the sense of Arkowitz and Lupton* [1].

Proof: First we show the following lemma:

Lemma 3.2. $\mathrm{cat}(X; K, L; A) \leq m$ *if and only if the inclusion $g : K \hookrightarrow X$ is compressible into $P^m(\Omega(X,L)) \subset P^\infty(\Omega(X,L)) \simeq X$ relative A as $\sigma : K \to P^m(\Omega(X,L))$ the structure map for* $\mathrm{cat}(X; K, L; A) \leq m$.

Proof: Let us assume that $\mathrm{cat}(X; K, L; A) \leq m$. Then by the definition of $\mathrm{cat}(X; K, L; A)$, the diagonal map $\Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X$ is compressible relative A into $\mathrm{T}^{m+1}(X,L)$. This implies that there exists a map σ from K to $P^m(\Omega(X,L))$, which is a compression relative A of the inclusion $g : K \hookrightarrow X$. Conversely, we assume that there is a compression relative A of the inclusion $g : K \hookrightarrow X$ into $P^m(\Omega(X,L))$. Composing with $q_m : P^m(\Omega(X,L)) \to \mathrm{T}^{m+1}(X,L)$, we obtain a compression relative A of the diagonal map $\Delta^{m+1}|_K : K \hookrightarrow X \to \prod^{m+1} X$ into $\mathrm{T}^{m+1}(X,L)$. □ □

Using this lemma, we obtain the following three propositions, which completes the proof of Theorem 3.1. □ □

Proposition 3.3 (Theorem 3.1 (1))**.** *Assume $X \supset K \supset L = A = *$. Then* $\mathrm{cat}(X; K, *; *) = \mathrm{cat}^{\mathrm{BG}}(X, K)$ *the relative L-S category in the sense of Berstein and Ganea* [2].

Proof: By Lemma 3.2 with $A = *$, $\mathrm{cat}(X; K, *; *) \leq m$ if and only if the inclusion $g : K \hookrightarrow X$ is compressible into $P^m(\Omega(X))$, which is equivalent with $\mathrm{cat}^{\mathrm{BG}}(X, K) \leq m$ by its definition. □ □

Proposition 3.4 (Theorem 3.1 (2))**.** *Assume $X = K \supset L = A \supset *$. Then $\mathrm{cat}(X; X, A; A) = \mathrm{cat}^{\mathrm{FH}}(X, A)$ the relative L-S category in the sense of Fadell and Husseini* [6].

Proof: By Lemma 3.2 with $X = K$ and $L = A$, $\mathrm{cat}(X; X, A; A) \leq m$ if and only if there is a right homotopy inverse of $e_m^{(X;X,A)} : P^m(\Omega(X, A)) \to X$ relative A, which is equivalent with $\mathrm{cat}^{\mathrm{FH}}(X, K) \leq m$ by its definition. □ □

Proposition 3.5 (Theorem 3.1 (3))**.** *Assume $h : X \to Y$ be a fibration with fibre $L \subset X$ and $K = X \supset L \supset A = *$. Then $\mathrm{cat}(X; X, L; *) = \mathrm{cat}^{\mathrm{AL}}(X, L)$ the relative L-S category in the sense of Arkowitz and Lupton* [1].

Proof: By Lemma 3.2 with $X = K$ and $A = *$, $\mathrm{cat}(X; X, L; *) \leq m$ if and only if there is a right homotopy inverse of $e_m^{(X;X,A)} : P^m(\Omega(X, A)) \to X$, which is equivalent with $\mathrm{cat}^{\mathrm{AL}}(X, L) \leq m$ by its definition. □ □

Among relative L-S categories, we state the relationship as follows:

Theorem 3.6. (1) *Let $(X; K, L; A)$ be a triad in $\mathcal{T}^A$. Then we obtain*

$$\mathrm{cat}(X; K, L; A) \leq \mathrm{cat}(X; K, A; A) \leq \mathrm{cat}(X; L, A; A) + \mathrm{cat}(X; K, L; A),$$
$$\mathrm{cat}(X; K, L; A) \leq \mathrm{cat}(X, L; A) \leq \mathrm{cat}(X, K; A) + \mathrm{cat}(X; K, L; A).$$

More generally, for any maps $f : (L, A) \to (X, A)$ and $g : (K, A) \to (X, A)$, we have

$$\mathrm{cat}(g, f) \leq \mathrm{cat}(g, *_A) \leq \mathrm{cat}(f, *_A) + \mathrm{cat}(g, f),$$
$$\mathrm{cat}(g, f) \leq \mathrm{cat}(1_{(X,A)}, f) \leq \mathrm{cat}(1_X, g) + \mathrm{cat}(g, f),$$

where $1_X : (X, A) = (X, A)$ denotes the identity and $*_A : (A, A) \hookrightarrow (X, A)$ denotes the trivial inclusion.

(2) *If $(X', L'; A) \supset (X, L; A)$ and $(K', A') \subset (K, A)$, then we have*

$$\begin{aligned}\mathrm{cat}(X'; K', L'; A') &\leq \mathrm{Min}\{\mathrm{cat}(X'; K, L'; A), \mathrm{cat}(X; K', L; A')\}\\ &\leq \mathrm{Max}\{\mathrm{cat}(X'; K, L'; A), \mathrm{cat}(X; K', L; A')\} \leq \mathrm{cat}(X; K, L; A).\end{aligned}$$

More generally, for any maps $f' : (L', A) \to (X', A)$, $f : (L, A) \to (X, A)$, $g : (K, A) \to (X, A)$, $h : (X, A) \to (X', A)$, $k : (K', A') \to (K, A)$ and $\ell : (L, A) \to (L', A)$, which satisfies the relation $f' \circ \ell = h \circ f$, we have

$$\begin{aligned}\mathrm{cat}(h \circ g \circ k, f') &\leq \mathrm{Min}\{\mathrm{cat}(h \circ g, f'), \mathrm{cat}(g \circ k, f)\}\\ &\leq \mathrm{Max}\{\mathrm{cat}(h \circ g, f'), \mathrm{cat}(g \circ k, f)\} \leq \mathrm{cat}(g, f).\end{aligned}$$

The following corollaries are immediate consequences of Theorem 3.6:

Corollary 3.7. (1) *For a triad $(X;K,L;*)$ in $\mathcal{T}_*$, we have*

$$\begin{aligned}\mathrm{cat}(X;K,L;*) &\le \mathrm{cat}(X;K) = \mathrm{cat}^{\mathrm{BG}}(X,K)\\ &\le \mathrm{cat}(X;L) + \mathrm{cat}(X;K,L;*) = \mathrm{cat}^{\mathrm{BG}}(X,L) + \mathrm{cat}(X;K,L;*),\\ \mathrm{cat}(X;K,L;*) &\le \mathrm{cat}(X,L;*) = \mathrm{cat}^{\mathrm{AL}}(X,L)\\ &\le \mathrm{cat}(X,K;*) + \mathrm{cat}(X;K,L;*) = \mathrm{cat}^{\mathrm{AL}}(X,K) + \mathrm{cat}(X;K,L;*).\end{aligned}$$

(2) *For a pair $(X,L;A)$ in $\mathcal{T}^A$, we have*

$$\begin{aligned}\mathrm{cat}(X,L;A) &\le \mathrm{cat}(X,A) = \mathrm{cat}^{\mathrm{FH}}(X,A)\\ &\le \mathrm{cat}(X;L,A) + \mathrm{cat}(X,L;A) \le \mathrm{cat}(X;L,A) + \mathrm{cat}^{\mathrm{FH}}(X,L).\end{aligned}$$

If we further assume that $A = *$, we have

$$\mathrm{cat}(X,L) \le \mathrm{cat}(X) \le \mathrm{cat}(X;L) + \mathrm{cat}(X,L).$$

(3) *For maps $f : L \subset X$, $f' : * \subset Y$, $g = 1_X : X \to X$, $h : X \to Y$, $k = 1_X : X \to X$ and $\ell : L \to *$ in $\mathcal{T}_*$ with $h|_L = \ell$, we have*

$$\mathrm{cat}^{\mathrm{BG}}(h) = \mathrm{cat}(h,*) = \mathrm{cat}(h\circ g, f') \le \mathrm{cat}(g,f) = \mathrm{cat}^{\mathrm{AL}}(X,L),$$

where $\mathrm{cat}^{\mathrm{AL}}(X,L)$ *must be denoted by* $\mathrm{cat}^{\mathrm{AL}}(h)$ *or even* $\mathrm{cat}(h)$*, if L is the fibre of a fibration h and we follow the notations in* [1].

Corollary 3.8. *In Definition 1.6, we have* $\mathrm{cat}(X;F_i,F_{i-1};A) \le 1$ *for the filtration* $\{F_i\}$*. Hence we have* $\mathrm{cat}(X;F_i,A;A) \le i$ *for every i.*

Proof: Proof of Theorem 3.6. The proofs for the general maps are left to the reader, and we concentrate ourselves to show the theorem for spaces.

Firstly, we show (1) for a triad $(X;K,L;A)$ in $\mathcal{T}^A$: To show $\mathrm{cat}(X;K,L;A) \le \mathrm{cat}(X;K,A;A)$, we assume that $\mathrm{cat}(X;K,A;A) = m$. By Lemma 3.2 for the triad $(X;K,A;A)$, $\mathrm{cat}(X;K,A;A) \le m$ if and only if there is a compression $\sigma : K \to P^m(\Omega(X,A))$ relative A of the inclusion $K \hookrightarrow X$. By Lemma 2.1 for the inclusion $(X;K,A;A) \hookrightarrow (X;K,L;A)$, the composition $P^m(\Omega(f|_{X,A}))\circ\sigma : K \to P^m(\Omega(X,L))$ gives the compression of the inclusion $K \hookrightarrow X$, which implies $\mathrm{cat}(X;K,L;A) \le m = \mathrm{cat}(X;K,A;A)$.

To show $\mathrm{cat}(X;K,L;A) \le \mathrm{cat}(X,L;A)$, we assume that $\mathrm{cat}(X,L;A) = m$. By Lemma 3.2 for the triad $(X;X,L;A)$, $\mathrm{cat}(X,L;A) \le m$ if and only if there is a compression $\sigma : X \to P^m(\Omega(X,L))$ relative A of the identity 1_X. By restricting σ to K, we obtain a compression $\sigma|_K : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$, which implies $\mathrm{cat}(X;K,L;A) \le m = \mathrm{cat}(X.L;A)$.

To show the inequality $\mathrm{cat}(X;K,A;A) \le \mathrm{cat}(X;L,A;A) + \mathrm{cat}(X;K,L;A)$, we assume that $\mathrm{cat}(X;L,A;A) = m$ and $\mathrm{cat}(X;K,L;A) = n$. By Lemma 3.2 for the triad $(X;L,A;A)$, $\mathrm{cat}(X;L,A;A) \le m$ if and only if there is a compression $\sigma : L \to P^m(\Omega(X,A))$ relative A of the inclusion $L \hookrightarrow X$. Then by Lemma 2.2 for the triad $(X;L,A;A)$, we have the following commutative ladder with $\sigma_0 = \sigma$ up to homotopy relative A:

$$\begin{tikzcd}
P^{n-1}(\Omega(X,L)) \arrow[r, hook] \arrow[d, "\sigma_{n-1}"] & P^{n}(\Omega(X,L)) \arrow[r, "e_n^{(X,L)}"] \arrow[d, "\sigma_n"] & X \arrow[d, "\mathrm{id}_X"] \\
P^{m+n-1}(\Omega(X,A)) \arrow[r, hook] & P^{m+n}(\Omega(X,A)) \arrow[r, "e_{m+n}^{(X,A)}"] & X.
\end{tikzcd}$$

Again by Lemma 3.2 for the triad $(X;K,L;L)$, $\mathrm{cat}(X;K,L;A) \le n$ if and only if there is a compression $\tau : K \to P^n(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Then the composition $\sigma_n \circ \tau : K \to P^{m+n}(\Omega(X,A))$ gives a compression relative A of the inclusion $K \hookrightarrow X$, which implies that $\mathrm{cat}(X;K,A;A) \le m+n = \mathrm{cat}(X;L,A;A) + \mathrm{cat}(X;K,L;A)$.

To show the inequality $\mathrm{cat}(X,L;A) \le \mathrm{cat}(X,K;A) + \mathrm{cat}(X;K,L;A)$, we assume that $\mathrm{cat}(X;K,L;A) = m$ and $\mathrm{cat}(X,K;A) = n$. By Lemma 3.2 for the triad $(X;K,L;A)$, $\mathrm{cat}(X;K,L;A) \le m$ if and only if there is a compression $\tau : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Then by Lemma 2.2 for the triad $(X;K,L;A)$, we have the following commutative ladder with $\tau_0 = \tau$ up to homotopy relative A:

$$\begin{tikzcd}
P^{n-1}(\Omega(X,K)) \arrow[r, hook] \arrow[d, "\tau_{n-1}"] & P^n(\Omega(X,K)) \arrow[r, "e_n^{(X,K)}"] \arrow[d, "\tau_n"] & X \arrow[d, "\mathrm{id}_X"] \\
P^{m+n-1}(\Omega(X,L)) \arrow[r, hook] & P^{m+n}(\Omega(X,L)) \arrow[r, "e_{m+n}^{(X,L)}"] & X.
\end{tikzcd}$$

Again by Lemma 3.2 for the triad $(X;X,K;A)$, $\mathrm{cat}(X,K;A) \le n$ if and only if there is a compression $\rho : X \to P^n(\Omega(X,K))$ relative A of the identity $1_X : X \to X$. Then the composition $\tau_n \circ \rho : X \to P^{m+n}(\Omega(X,L))$ gives a compression relative A of the identity $1_X : X \to X$, which implies that $\mathrm{cat}(X,L;A) \le m+n = \mathrm{cat}(X,K;A) + \mathrm{cat}(X;K,L;A)$.

Secondly, we show (2) for a triad $(X;K,L;A)$ with spaces $X' \supset X$, $(K',A') \subset (K,A)$ and $(L',A') \subset (L,A)$. It is sufficient to show that $\mathrm{cat}(X';K,L';A) \le \mathrm{cat}(X;K,L;A)$ and $\mathrm{cat}(X;K',L;A') \le \mathrm{cat}(X;K,L;A)$:

To show $\mathrm{cat}(X';K,L';A) \le \mathrm{cat}(X;K,L;A)$, we assume that $\mathrm{cat}(X;K,L;A) = m$. By Lemma 3.2 for the triad $(X;K,L;A)$, $\mathrm{cat}(X;K,L;A) \le m$ if and only if there is a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Since $X' \supset X$, we have the inclusion of triads : $(X;K,L;A) \hookrightarrow (X';K,L';A)$. Then by Lemma 2.1 for the map of triads $j : (X;K,L;A) \hookrightarrow (X';K,L';A)$, we have the following commutative ladder up to homotopy relative A:

$$\begin{tikzcd}
P^{m-1}(\Omega(X,L)) \arrow[r, hook] \arrow[d, "j_{m-1}"] & P^m(\Omega(X,L)) \arrow[r, "e_m^{(X,L)}"] \arrow[d, "j_m"] & X \arrow[d, hook, "j|_X"] \\
P^{m-1}(\Omega(X',L')) \arrow[r, hook] & P^m(\Omega(X',L')) \arrow[r, "e_m^{(X',L')}"] & X'
\end{tikzcd}$$

with $j_0 = \mathrm{id}_L$ and $j_k = P^k(\Omega(j|_{(X,L)}))$, $1 \le k \le m$. Thus the map $j_m \circ \sigma$ gives a compression relative A of the inclusion $K \hookrightarrow X \subset X'$, and hence $\mathrm{cat}(X';K,L';A)$ $\le m = \mathrm{cat}(X;K,L;A)$.

To show $\mathrm{cat}(X;K',L;A') \le \mathrm{cat}(X;K,L;A)$, we may assume that $A = A'$, since it is clear by definition that $\mathrm{cat}(X;K,L;A') \le \mathrm{cat}(X;K,L;A)$ if $A' \subset A$: let us assume that $\mathrm{cat}(X;K,L;A) = m$. By Lemma 3.2 for the triad $(X;K,L;A)$, $\mathrm{cat}(X;K,L;A) \le m$ if and only if there is a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$. Hence the restriction $\sigma|_{K'}$ of the map σ to K' gives a compression relative A of the inclusion $K' \hookrightarrow X$, and hence $\mathrm{cat}(X;K',L;A) \le m = \mathrm{cat}(X;K,L;A)$. $\square$ $\square$

4. A higher Hopf invariant for a triad

Let us consider the following exact sequences of abelian groups and algebraic loops:

$$0 \to [\Sigma V, E^{m+1}(\Omega(X,L))] \xrightarrow{p^{(X,L)}_{m+1*}} [\Sigma V, P^m(\Omega(X,L))] \xrightarrow{e^{(X,L)}_{m*}} [\Sigma V, X] \to 0 \tag{4.1}$$

$$1 \to [V, E^{m+1}(\Omega(X,L))] \xrightarrow{p^{(X,L)}_{m+1*}} [V, P^m(\Omega(X,L))] \xrightarrow{e^{(X,L)}_{m*}} [V, X]. \tag{4.2}$$

Since the fibre $\Omega(X)$ of the fibration $p^{(X,L)}_{m+1}$ is contractible in the total space $E^{m+1}(\Omega(X,L))$ of $p^{(X,L)}_{m+1}$, we know $e^{(X,L)}_{m*} : [\Sigma V, P^m(\Omega(X,L))] \to [\Sigma V, X]$ is an epimorphism of abelian groups and $p^{(X,L)}_{m+1*} : [\Sigma V, E^{m+1}(\Omega(X,L))] \to [\Sigma V, P^m(\Omega(X,L))]$ is a monomorphism of abelian groups. Similarly, $p^{(X,L)}_{m+1*} : [V, E^{m+1}(\Omega(X,L))] \to [V, P^m(\Omega(X,L))]$ is a monomorphism of algebraic loops. Thus we obtain the following proposition:

Proposition 4.1. (1) *$e^{(X,L)}_{m*} : [\Sigma V, P^m(\Omega(X,L))] \to [\Sigma V, X]$ is an epimorphism of abelian groups.*
(2) *$p^{(X,L)}_{m+1*} : [\Sigma V, E^{m+1}(\Omega(X,L))] \to [\Sigma V, P^m(\Omega(X,L))]$ is a monomorphism of abelian groups.*
(3) *$p^{(X,L)}_{m+1*} : [V, E^{m+1}(\Omega(X,L))] \to [V, P^m(\Omega(X,L))]$ is a monomorphism of algebraic loops.*

We give here a definition of Higher Hopf invariants in a slightly different form as follows:

Definition 4.2. (1) *Let $(X;K,L;A)$ be a triad in $\mathcal{T}^A$, V be a co-loop co-H-space, and $\alpha : V \to K$ a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. We assume that $\mathrm{cat}(X;K,L;A) \leq m$. By Lemma 3.2 for the triad $(X;K,L;A)$, $\mathrm{cat}(X;K,L;A) \leq m$ implies that the inclusion $i : K \hookrightarrow X$ is compressible into $P^m(\Omega(X,L))$ relative A as a map $\sigma : K \to P^m(\Omega(X,L))$. Since $e^{(X,L)}_m \circ\sigma\circ\alpha \sim i\circ\alpha$ is trivial in $\hat{K} \subset X$, we obtain a unique lift $H^\sigma_m(\alpha) : V \to E^{m+1}(\Omega(X,L)) \simeq \Omega(X,L) * \Omega(X) * \underset{m}{\cdots} * \Omega(X)$ of $\sigma\circ\alpha$. We define $H^{(X;K,L;A)}_m(\alpha)$ as follows:*

$$H^{(X;K,L;A)}_m(\alpha) = \left\{ [H^\sigma_m(\alpha)] \;\middle|\; \begin{array}{l} \sigma : K \to P^m(\Omega(X,L)) \textit{ is a compression rela-} \\ \textit{tive } A \textit{ of the inclusion } K \hookrightarrow X. \end{array} \right\}$$
$$\subset [V, \Omega(X,L) * \Omega(X) * \underset{m}{\cdots} * \Omega(X)].$$

(2) *Let $(K,L;A)$ be a pair in $\mathcal{T}^A$ and let $\alpha : V \to K$ a map in $\mathcal{T}$. We assume that $\mathrm{cat}(K,L;A) \leq m$. By Lemma 3.2 for the triad $(K;K,L;A)$, $\mathrm{cat}(K,L;A) \leq m$ implies that the identity $1_K : K \to K$ is compressible into $P^m(\Omega(K,L))$ relative A as a map $\sigma : K \to P^m(\Omega(K,L))$. By Lemma 2.1 for the inclusion $j : (K;K,*;*) \hookrightarrow (K;K,L;*)$, the following ladder is commutative up to homotopy:*

$$\begin{tikzcd}
* \arrow[r, hook] \arrow[d, hook] & \Sigma\Omega(K) \arrow[r, hook] \arrow[d, hook, "j_1"] & P^m(\Omega(K)) \arrow[r, "e^K_m"] \arrow[d, hook, "j_m"] & K \arrow[d, "\mathrm{id}_K"'] \\
L \arrow[r, hook] & P^1(\Omega(K,L)) \arrow[r, hook] & P^m(\Omega(K,L)) \arrow[r, "e^{(K,L)}_m"'] & K,
\end{tikzcd}$$

where $e_1^K = e_m^K|_{\Sigma\Omega(K)} : \Sigma\Omega(K) \to K$ is given by the evaluation map (see Ganea [8] or [12]). Since V is a co-loop co-H-space, the evaluation map $e_1^V : \Sigma\Omega(V) \to V$ admits a right homotopy inverse, say the co-H-structure map $\rho^V : V \to \Sigma\Omega(V)$ for V, by Ganea [9]. Then we have $e_1^K \circ \Sigma\Omega(\alpha) \circ \rho^V \sim \alpha \circ e_1^V \circ \rho^V \sim \alpha$, and hence $e_1^{(K,L)} \circ j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V \simeq \mathrm{id}_K \circ\, e_1^K \circ \Sigma\Omega(\alpha) \circ \rho^V \sim \alpha$. Since both the maps $e_1^{(K,L)} \circ \sigma \circ \alpha$, $e_1^{(K,L)} \circ \sigma \circ \alpha$ and $e_1^{(K,L)} \circ j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V$ are homotopic to α, the difference $d(\alpha) = \sigma \circ \alpha - j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V$ is trivial in K. Thus we obtain a unique lift $H_m^\sigma(\alpha) : V \to E^{m+1}(\Omega(K,L)) \simeq \Omega(X,L) * \Omega(X) * \underset{m}{\cdots} * \Omega(X)$ of $d(\alpha)$. We define $H_m^{(K,L;A)}(\alpha)$ as follows:

$$\begin{aligned} H_m^{(K,L;A)}(\alpha) &= \left\{ [H_m^\sigma(\alpha)] \;\middle|\; \begin{array}{l} \sigma \text{ is a compression relative } A \text{ of the identity} \\ 1_K. \end{array} \right\} \\ &\subset [V, \Omega(K,L) * \Omega(K) * \underset{m}{\cdots} * \Omega(K)]. \end{aligned}$$

We then show the following result which clarifies how a higher Hopf invariant determines whether a cone decomposition reduces to a categorical sequence or not.

Theorem 4.3. *Let $(X;K,L;A)$ be a triad in $\mathcal{T}^A$, V be a co-loop co-H-space and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. If $\mathrm{cat}(X;K,L;A) \le m$ and $H_m^{(X;K,L;A)}(\alpha) = 0$, then $\mathrm{cat}(X;\hat{K},L;A) \le m$.*

Proof: Let $(X;K,L;A)$ be a triad in $\mathcal{T}^A$, V be a co-loop co-H-space and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. Assuming $\mathrm{cat}(X;K,L;A) \le m$ and $H_m^{(X;K,L;A)}(\alpha) = 0$, we show $\mathrm{cat}(X;\hat{K},L;A) \le m$: by the assumption, there is a compression $\sigma : K \to P^m(\Omega(X,L))$ relative A of the inclusion $K \hookrightarrow X$ such that $\sigma \circ \alpha \sim p_{m+1}^{(X,L)} \circ H_m^\sigma(\alpha) \sim *$, and hence there is a map $\hat{\sigma} : \hat{K} \to P^m(\Omega(X,L))$ whose restriction to K is σ. Since $e_m^{(X,L)} \circ \sigma$ and the inclusion $K \hookrightarrow X$ are homotopic relative A, the difference between $e_m^{(X,L)} \circ \hat{\sigma}$ and the inclusion $\hat{K} \hookrightarrow X$ is given by an element $[\delta] \in [\Sigma V, X]$. By Proposition 4.1 (1), we have a map $\hat{\delta} : \Sigma V \to P^m(\Omega(X,L))$ such that $e_m^{(X,L)} \circ \hat{\delta} \sim \delta$. By subtracting $\hat{\delta}$ from $\hat{\sigma}$, we obtain a genuine compression $\sigma' = \hat{\sigma} - \hat{\delta} : \Sigma V \to P^m(\Omega(X,L))$ of the inclusion $\hat{K} \to P^m(\Omega(X,L))$ relative A, where the subtraction is given by the co-action of ΣV under $K \cup_\alpha C^2 V = \hat{K}$ the map cone of α. This implies that $\mathrm{cat}(X;\hat{K},L;A) \le m$. $\square$ $\square$

We describe here the relationship among higher Hopf invariants. The following definition is essentially due to Berstein and Hilton [3]:

Definition 4.4. *Let $(X;K,L;A)$ and $(X';K',L';A)$ be triads in $\mathcal{T}^A$, V be a co-loop co-H-space, and $s : K \to \mathrm{T}^{m+1}(X,L)$ and $s' : K' \to \mathrm{T}^{m+1}(X',L')$ be compressions of $\Delta^{m+1} \circ i : K \hookrightarrow \prod^{m+1} X$ and $\Delta^{m+1} \circ i' : K' \hookrightarrow \prod^{m+1} X'$ relative A, respectively, so that $\mathrm{cat}(X;K,L;A) \le m$ and $\mathrm{cat}(X';K',L';A) \le m$. A map $f : (X;K,L;A) \to (X';K',L';A)$ of triads in $\mathcal{T}^A$ is called m-primitive (with respect to s and s'), if $s' \circ f|_K \sim \mathrm{T}^{m+1}(f|_{(X',L')}) \circ s$.*

Let $(X;K,L;A)$ and $(X';K',L';A)$ be triads in $\mathcal{T}^A$, and let $\mathrm{cat}(X;K,L;A) \le m$ and $\mathrm{cat}(X';K',L';A) \le m$ with compressions $s : K \to \mathrm{T}^{m+1}(X,L)$ and $s' : K' \to \mathrm{T}^{m+1}(X',L')$ of $\Delta^{m+1} \circ i : K \hookrightarrow \prod^{m+1} X$ and $\Delta^{m+1} \circ i' : K' \hookrightarrow \prod^{m+1} X'$ relative A, respectively. By using the lower right square of the diagram (2.2), we obtain structure maps σ, σ' for $\mathrm{cat}(X;K,L;A) \le m$ and $\mathrm{cat}(X';K',L';A) \le m$

corresponding to s and s', respectively by $s \sim q_m^{(X,L)}\circ\sigma$ and $s' \sim q_m^{(X',L')}\circ\sigma'$ relative A.

Lemma 4.5. *Let $f : (X;K,L;A) \to (X';K',L';A)$ be a map of triads in $\mathcal{T}^A$. Then f is m-primitive with respect to s and s', if and only if $\sigma'\circ f|_K \sim P^m(\Omega(f|_{(X,L)}))\circ\sigma$ relative A for the corresponding structure maps σ and σ'.*

Proof: Assume that f satisfies that $\sigma'\circ f|_K \sim P^m(\Omega(f|_{(X,L)}))\circ\sigma$. By composing $q_m^{(X',L')} : P^m(\Omega(X',L')) \to \mathrm{T}^{m+1}(X',L')$ with the both sides, we obtain

$$\begin{aligned} s'\circ f|_K &\sim q_m^{(X',L')}\circ\sigma'\circ f|_K \\ &\sim q_m^{(X',L')}\circ P^m(\Omega(f|_{(X,L)}))\circ\sigma \\ &\sim \mathrm{T}^{m+1}(f|_{(X,L)})\circ q_m^{(X,L)}\circ\sigma \\ &\sim \mathrm{T}^{m+1}(f|_{(X,L)})\circ s \end{aligned}$$

relative A, and hence f is m-primitive with respect to s and s'. Conversely assume that f is m-primitive with respect to s and s'. Then the naturality of the lower right square of the diagram (2.2) immediately induces the homotopy relation $\sigma'\circ f|_K \sim P^m(\Omega(f|_{(X,L)}))\circ\sigma$ relative A. □ □

Theorem 4.6. *Let $(X;K,L;A)$ and $(X';K',L';A)$ be triads in $\mathcal{T}^A$, V be a co-loop co-H-space, and $s : K \to \mathrm{T}^{m+1}(X,L)$ and $s' : K' \to \mathrm{T}^{m+1}(X',L')$ be compressions of the inclusions $i : K \hookrightarrow X$ and $i' : K' \hookrightarrow X'$ relative A, respectively, so that $\mathrm{cat}(X;K,L;A) \le m$ and $\mathrm{cat}(X';K',L';A) \le m$, respectively. Let $f : (X;K,L;A) \to (X';K',L';A)$ be a map of triads in $\mathcal{T}^A$ and let $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$ and $X' \supset \hat{K}' = K' \cup_{f|_K\circ\alpha} CV \supset K$. If f is m-primitive with respect to s and s', then we have*

$$E^{m+1}(\Omega(f|_{(K,L)}))_\#\circ H_m^{(X;K,L;A)}(\alpha) \subset H_m^{(X';K',L';A)}(f|_K\circ\alpha).$$

Proof: By Lemma 2.1 for $f : (X;K,L;A) \to (X';K',L';A)$ a map of triads in $\mathcal{T}^A$, the following diagram is commutative up to homotopy relative A:

$$\begin{CD} E^{m+1}(\Omega(X,L)) @>{p_m^{\Omega(X,L)}}>> P^m(\Omega(X,L)) @>{e_m^{(X,L)}}>> X \\ @V{E^{m+1}(\Omega(f|_{(X,L)}))}VV @V{P^m(\Omega(f|_{(X,L)}))}VV @VV{f|_X}V \\ E^{m+1}(\Omega(X',L')) @>>{p_m^{\Omega(X',L')}}> P^m(\Omega(X',L')) @>>{e_m^{(X',L')}}> X'. \end{CD}$$

Since f is m-primitive with respect to s and s', we have the homotopy relation relative A $P^m(\Omega(f|_{(X,L)}))\circ\sigma \sim \sigma'\circ f|_K$ for the corresponding compressions σ and σ' relative A of the inclusions $i : K \hookrightarrow X$ and $i' : K' \hookrightarrow X'$, resp. Thus we have the following homotopy relation:

$$\begin{aligned} &p_m^{\Omega(X',L')}\circ E^{m+1}(\Omega(f|_{(X,L)}))\circ H_m^\sigma(\alpha) \\ &\quad\sim P^m(\Omega(f|_{(X,L)}))\circ p_m^{\Omega(X,L)}\circ H_m^{(X;K,L;A)}(\alpha) \\ &\quad\sim P^m(\Omega(f|_{(X,L)}))\circ\sigma\circ\alpha \sim \sigma'\circ f|_K\circ\alpha \sim p_m^{\Omega(X',L')}\circ H_m^{\sigma'}(f|_K\circ\alpha). \end{aligned}$$

Hence we obtain $E^{m+1}(\Omega(f|_{(X,L)}))\circ H_m^\sigma(\alpha) \sim H_m^{\sigma'}(f|_K\circ\alpha)$, since $p_{m\,*}^{\Omega(X',L')}$ is monic by Proposition 4.1 (3). Thus we have $E^{m+1}(\Omega(f|_{(X,L)}))_\# H_m^{(X;K,L;A)}(\alpha) \subset H_m^{(X';K',L';A)}(f|_K\circ\alpha)$. □ □

Theorem 4.7. *Let $(X, K, L; A)$ be a triple in $\mathcal{T}^A$, V be a co-loop co-H-space, and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. If* $\operatorname{cat}(K, L; A) \leq m$, *then we have*

$$E^{m+1}(\Omega(j|_{(K,L)}))_\# \circ H_m^{(K,L;A)}(\alpha) \subset H_m^{(X;K,L;A)}(\alpha),$$

where $j : (K; K, L; A) \to (X; K, L; A)$ is the inclusion.

Corollary 4.8. *For the filtration $\{F_i\}$ in Definition 1.6, we have*

$$E^{m+1}(\Omega(j_i|_{(F_i,F_{i-1})}))_\# \circ H_i^{(F_i,F_{i-1};A)}(\alpha) \subset H_i^{(X;F_i,F_{i-1};A)}(\alpha)$$

for every i, where $j_i : (F_i; F_i, F_{i-1}; A) \hookrightarrow (X; F_i, F_{i-1}; A)$ denote the inclusion.

Proof: Proof of Theorem 4.7 Let $(X, K, L; A)$ be a triple in $\mathcal{T}^A$, V be a co-loop co-H-space and $\alpha : V \to K$ be a map in $\mathcal{T}$ such that $X \supset \hat{K} = K \cup_\alpha CV \supset K$. Assuming $\operatorname{cat}(K, L; A) \leq m$, we show $E^{m+1}(\Omega(j|_{(K,L)}))_\# H_m^{(K,L;A)}(\alpha) \subset H_m^{(X;K,L;A)}(\alpha)$, where $j : (K; K, L; A) \to (X; K, L; A)$ denotes the inclusion: By Lemma 2.1 for $j : (K; K, L; A) \to (X; K, L; A)$ an inclusion map of triads in $\mathcal{T}^A$, the following diagram is commutative up to homotopy relative A:

$$\begin{tikzcd}
E^{m+1}(\Omega(K,L)) \arrow[r, "p_m^{\Omega(K,L)}"] \arrow[d, "E^{m+1}(\Omega(j|_{(K,L)}))"'] & P^m(\Omega(X,L)) \arrow[r, "e_m^{(K,L)}"] \arrow[d, "P^m(\Omega(j|_{(K,L)}))"'] & K \arrow[d, hook, "j|_K"] \\
E^{m+1}(\Omega(X,L)) \arrow[r, "p_m^{\Omega(X,L)}"'] & P^m(\Omega(X,L)) \arrow[r, "e_m^{(X,L)}"'] & X.
\end{tikzcd}$$

By the definition of a higher Hopf invariant, we obtain $p_m^{\Omega(K,L)} \circ H_m^\sigma(\alpha) \sim \sigma \circ \alpha - j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V$, and hence we have the homotopy relation

$$\begin{aligned}
p_m^{\Omega(X,L)} \circ E^{m+1}(\Omega(j|_{(K,L)})) \circ H_m^\sigma(\alpha) &\sim P^m(\Omega(j|_{(K,L)})) \circ p_m^{\Omega(K,L)} \circ H_m^\sigma(\alpha) \\
&\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - P^m(\Omega(j|_{(K,L)})) \circ j_1 \circ \Sigma\Omega(\alpha) \circ \rho^V \\
&\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - j_1 \circ \Sigma\Omega(j|_{(K,L)}) \Sigma\Omega(\alpha) \circ \rho^V \\
&\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha - j_1 \circ \Sigma\Omega(j|_{(K,L)} \circ \alpha) \circ \rho^V \\
&\sim P^m(\Omega(j|_{(K,L)})) \circ \sigma \circ \alpha,
\end{aligned}$$

since $j|_{(K,L)} \circ \alpha \sim *$ in X. This implies that $E^{m+1}(\Omega(j|_{(K,L)})) \circ H_m^\sigma(\alpha)$ is homotopic to $H_m^{P^m(\Omega(j|_{(K,L)})) \circ \sigma}(\alpha)$, and hence $E^{m+1}(\Omega(j|_{(K,L)}))_\# \circ H_m^{(K,L;A)}(\alpha) \subset H_m^{(X;K,L;A)}(\alpha)$. □

5. Categorical sequence

Let F_i^X, $0 \leq i \leq m$, and F_j^Y, $0 \leq j \leq n$, be categorical sequences for $(X, A) \in \mathcal{T}^A$ and $(Y, A) \in \mathcal{T}^A$, respectively. Then for a map $f : (X, A) \to (Y, A)$, we say that f preserves categorical sequences, if $f(F_i^X) \subset F_i^Y$ for all $i \geq 0$. We first show the following:

Lemma 5.1. *Let $(X, A) \in \mathcal{T}^A$ be dominated by $(Y, A) \in \mathcal{T}^A$ with a categorical sequence of length m. Then there is a categorical sequence for (X, A) of length m compatible with the given categorical sequence for (Y, A), i.e., the inclusion $i : (X, A) \hookrightarrow (Y, A)$ and the retraction $r : (Y, A) \to (X, A)$ preserve categorical sequences.*

The above lemma yields the relation between the L-S category and the length of a categorical sequence:

Theorem 5.2. *For any* X *in* $\mathcal{T}$*, we have* $\mathrm{cat}(X) = \mathrm{catlen}(X)$*. More generally, for any object* $(X, A) \in \mathcal{T}^A$*, we have* $\mathrm{cat}^{\mathrm{FH}}(X, A) = \mathrm{catlen}(X, A)$*.*

Proof: Assume $\mathrm{catlen}(X, A) = m$ with a categorical sequece (F_i^X, A), $0 \le i \le m$ for (X, A). Then by Corollary 3.8, we have $\mathrm{cat}(X, A) = \mathrm{cat}(X; X, A) = \mathrm{cat}(X; F_m^X, A)$ $\le m = \mathrm{catlen}(X, A)$. Hence we have $\mathrm{cat}(X, A) \le \mathrm{catlen}(X, A)$. Conversely assume $\mathrm{cat}(X, A) = m$. Then the pair (X, A) is dominated by $(P^m(\Omega(X, A)), A)$ which has the cone decomposition $(P^i(\Omega(X, A)), A)$, $0 \le i \le m$ as the canonical categorical sequence. Thus by Lemma 5.1, we have that (X, A) has also a categorical sequence of length m, and hence that $\mathrm{catlen}(X, A) \le m = \mathrm{cat}(X, A)$. It completes the proof of Theorem 5.2. ☐

Proof: Proof of Lemma 5.1. Let (F_i^Y, A), $0 \le i \le m$, be a categorical sequence for $(Y, A) \in \mathcal{T}^A$ and $\sigma : X \to Y$ and $\rho : Y \to X$ be maps such that $\rho \circ \sigma \sim 1_X$. Then we define F_i as the homotopy pullback of σ and the inclusion $\iota_i : F_i^Y \hookrightarrow F_m^Y$. Since the image of $\sigma|_A$ is the same as the inclusion $A \subseteq F_0^Y \hookrightarrow F_m^Y$, the space A is canonically embedded in F_0 and hence in $F_i \supset F_0$ for any $i \ge 0$.

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & F_0 \arrow[r, hook, "\hat{\iota}_0"] \arrow[d, "\sigma_0"] \arrow[dr, phantom, "\mathrm{PB}"] & F_i \arrow[r, hook, "\hat{\iota}_i"] \arrow[d, "\sigma_i"] \arrow[dr, phantom, "\mathrm{PB}"] & F_m \arrow[r, "\hat{\mathrm{id}}"] \arrow[d, "\sigma_m"] \arrow[dr, phantom, "\mathrm{HPB}"] & X \arrow[d, "\sigma"] \\
* \arrow[r, hook] & A \arrow[r, hook, "\iota_0"'] & F_i^Y \arrow[r, hook, "\iota_i"'] & F_m^Y \arrow[r, equal, "\mathrm{id}"'] & F_m^Y,
\end{tikzcd}$$

where F denotes the homotopy fibre of σ and F_m is the homotopy pullback of σ and the identity of F_m^Y. Since $\rho \circ \sigma \sim 1_X$, $\rho|_{F_i^Y}$ can be compressed into F_i and we have the following commutative diagram:

$$\begin{tikzcd}
* \arrow[r, hook] \arrow[d] & A \arrow[r, hook, "\iota_0"'] \arrow[d, "\rho_0"'] & F_i^Y \arrow[r, hook, "\iota_i"'] \arrow[d, "\rho_i"'] & F_m^Y \arrow[r, equal, "\mathrm{id}"'] \arrow[d, "\rho_m"'] & F_m^Y \arrow[d, "\rho"] \\
F \arrow[r, hook] & F_0 \arrow[r, hook, "\hat{\iota}_0"] & F_i \arrow[r, hook, "\hat{\iota}_i"] & F_m \arrow[r, "\hat{\mathrm{id}}"] & X.
\end{tikzcd}$$

Then by the definition of categorical sequence, there is a compression $\nu_i^Y : F_i^Y \to F_m^Y \times * \cup F_{i-1}^Y \times F_m^Y$ of the diagonal map $\Delta_{F_i^Y} : F_i^Y \to F_i^Y \times F_i^Y \subseteq F_m^Y \times F_m^Y$ relative to F_{i-1}^Y:

$$\begin{tikzcd}
F_i^Y \arrow[r, "\Delta_{F_i^Y}"] \arrow[drr, "\nu_i^Y"'] & F_i^Y \times F_i^Y \arrow[rr, hook] & & F_m^Y \times F_m^Y \\
& & F_m^Y \times * \cup F_{i-1}^Y \times F_m^Y \arrow[ur, hook] &
\end{tikzcd}$$

By composing ρ_i and σ_i, we obtain a compression of the diagonal map $\Delta_{F_i} : F_i \to F_i{\times}F_i \subseteq F_m{\times}F_m$ as follows:

$$\begin{tikzcd}
 & & F_m{\times}{*} \cup F_{i-1}{\times}F_m \arrow[dr, hook] & \\
F_i \arrow[r, "\Delta_{F_i}"] \arrow[urr, dotted, "\nu_i"] \arrow[d, "\sigma_i"] & F_i{\times}F_i \arrow[rr, hook] & & F_m{\times}F_m \\
F_i^Y \arrow[r, "\Delta_{F_i^Y}"] \arrow[dr, "\nu_i^Y"'] & F_i^Y{\times}F_i^Y \arrow[rr, hook] & & F_m^Y{\times}F_m^Y \arrow[u, "\rho_m{\times}\rho_m"'] \\
 & F_m^Y{\times}{*} \cup F_{i-1}^Y{\times}F_m^Y \arrow[uuu, "\rho_{m,i-1}"] \arrow[urr, hook] & &
\end{tikzcd}$$

This implies $\mathrm{cat}(X';X',F_{m-1}^X;A) \leq 1$, and hence $X' = F_m^X \supset F_{m-1} \supset \cdots \supset F_0 = A$ gives a categorical sequence for X. □ □

The following lemma is our version of the result of Arkowitz and Lupton [1]:

Lemma 5.3. *Let X be a space in $\mathcal{T}$ with $\mathrm{cat}(X) = m$ and $\{F_i\,;\, 0{\leq}i{\leq}m\}$ be a categorical sequence for X. Then there is a map $\mu : F_i \to F_m/F_{i-1} \vee F_m$ in $\mathcal{T}$ with axes $F_i \to F_m/F_{i-1}$ and the inclusion $F_i \hookrightarrow F_m$.*

Proof: By the definition of a categorical sequence, the diagonal map $\Delta : F_i \to F_i{\times}F_i \subseteq F_m{\times}F_m$ is compressible into $F_{i-1}{\times}F_m \cup F_m{\times}{*}$ as $F_i \xrightarrow{\hat{\mu}} F_{i-1}{\times}F_m \cup F_m{\times}{*} \subseteq F_m{\times}F_m$. Since $F_m/F_{i-1} \vee F_m$ can be regarded as the pushout of the second projection $\mathrm{pr}_2 : F_{i-1}{\times}F_m \to F_m$ and the canonical inclusion $\iota : F_{i-1}{\times}F_m \hookrightarrow F_{i-1}{\times}F_m \cup F_m{\times}{*}$, we have the following diagram:

$$\begin{tikzcd}
 & F_i \arrow[d, "\hat{\mu}"'] \arrow[dr, "\Delta"] & \\
F_{i-1}{\times}F_m \arrow[r, hook] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\mathrm{PO}"] & F_{i-1}{\times}F_m \cup F_m{\times}{*} \arrow[r, hook] \arrow[d, "\hat{q}_i^{F_i}"'] \arrow[dr, phantom, "\mathrm{PO}"] & F_m{\times}F_m \arrow[d, "q_i^{F_i}\times\mathrm{id}_{F_i}"] \\
F_m \arrow[r, hook, "\mathrm{in}_2"] & F_m/F_{i-1} \vee F_m \arrow[r, hook, "j"] & F_m/F_{i-1} \times F_m,
\end{tikzcd}$$

where $q_i^{F_i} : F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$ denotes the canonical collapsing map in $\mathcal{T}$. Let μ be the composition $\hat{q}_i^{F_i}{\circ}\hat{\mu} : F_i \to F_i/F_{i-1} \vee F_m$ so that $j{\circ}\mu$ is homotopic to $(q_i^{F_i}{\times}\,\mathrm{id}_{F_i}){\circ}\Delta$. Thus μ has axes $q_i^{F_i} : F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$ and the inclusion $F_i \hookrightarrow F_m$. □ □

Corollary 5.4. *Let (X,A) be an object in $\mathcal{T}^A$. If $\mathrm{cat}^{\mathrm{FH}}(X,A) = m > 0$, then there exists a sequence for pairs $\{(F_i,A)\,;\, 0{\leq}i{\leq}m\}$ such that $(F_0,A) \simeq (A,A)$ in*

(F_m, A), $(F_m, A) \simeq (X, A)$ relative A and $\mathrm{cat}(X; F_i, A) \le i$, $i > 0$. Moreover we have $\mathrm{cat}(F_m/F_{i-1}; F_i/F_{i-1}) \le 1$ with a 'partial co-action' $F_i \to F_m/F_{i-1} \vee F_m$ along the collapsion $F_i \to F_i/F_{i-1} \subseteq F_m/F_{i-1}$, $i > 0$. In particular, F_m/F_{m-1} is a co-H-space coacting on F_m along the collapsion $F_m \to F_m/F_{m-1}$.

6. Examples of categorical sequences

In [3], Berstein and Hilton showed that the L-S category of the cell complex $Q(\alpha) = S^r \cup_\alpha e^{q+1}$, $\alpha \in \pi_q(S^r)$ is determined by the Hopf invariant $H_1(\alpha) \in \pi_{q+1}(S^r \times S^r, S^r \vee S^r)$ ($\cong \pi_q(\Omega(S^r) * \Omega(S^r))$ by Ganea). We can easily observe that $F_0 = *$, $F_1 = S^r$ and $F_2 = Q(\alpha)$ give a cone decomposition of $Q(\alpha)$ of length 2. If $H_1(\alpha) = 0$, then by Theorem 4.3, we obtain that $F'_0 = F_0 = *$, $F'_1 = F_1 \cup_\alpha e^{q+1} = F_2 = Q(\alpha)$ give a categorical sequence of length 1.

In [13], the author showed that the L-S category of total space $E(\beta) = Q(\beta) \cup_{\psi(\beta)} e^{q+r+1}$, $\beta \in \pi_q(S^r)$, $\psi(\beta) \in \pi_{q+r}(Q(\beta))$ is determined by $\Sigma^r H_1(\beta) \in \pi_{q+r}(\Omega(S^r) * \Omega(Q(\beta)) * \Omega(Q(\beta)))$, if $H_1(\beta) \neq 0$. We can easily observe that $F_0 = *$, $F_1 = S^r$, $F_2 = Q(\beta)$ and $F_3 = E(\beta)$ give a cone decomposition of $E(\beta)$ of length 3. If $\Sigma^r H_1(\alpha) = 0$, then by Theorem 4.3, we obtain that $F'_0 = F_0 = *$, $F'_1 = F_1 = S^r$, $F'_2 = F_2 \cup_{\psi(\beta)} e^{q+r+1} = F_3 = E(\beta)$ give a categorical sequence of length 2.

In [15], Kono and the author showed that there is a cone decomposition E_i, $0 \le i \le 8$ and E'_8 of $\mathbf{Spin}(9)$ of length 9, while the L-S category of $\mathbf{Spin}(9)$ is 8 by a combination of a higher Hopf invariant and the cone decomposition: We can easily see that Lemma 1.1 in [15] implies that the higher Hopf invariant of the attaching map of the top cell of $\mathbf{Spin}(9)$ must vanish, since the structure map of $\mathrm{cat}(E'_8) = 8$ can be chosen to be compatible to the structure map of $\mathrm{cat}(E_8) = 8$ by the argument given in the proof of Lemma 1.1 in [15]. Hence by Theorem 4.3, we obtain that E_i, $0 \le i \le 7$ and E'_8 give the categorical sequence of length 8.

7. Cup length and Module weight for the relative L-S category

A computable lower estimate is given by the classical cup-length. Here we give the definition for our new relative L-S category.

Definition 7.1. *For any two maps $f : (L, A) \subset (X, A)$ and $g : (K, A) \to (X, A)$ in $\mathcal{T}^A$, we define cup length for $(g, f) = (X; K, L; A)$*

1. *Let h be a multiplicative generalized cohomology theory.*
$$\mathrm{cup}(g, f; h) = \mathrm{Min}\left\{ m \ge 0 \,\middle|\, \begin{array}{l} \forall \{v_0 \in h^*(X, L); v_1, \cdots, v_m \in h^*(X, A)\} \\ g^*(v_0 \cdot v_1 \cdots v_m) = 0 \text{ in } h^*(K, A) \end{array} \right\}.$$
2. $$\mathrm{cup}(g, f) = \mathrm{Max}\left\{ \mathrm{cup}(g, f; h) \,\middle|\, \begin{array}{l} h \text{ is a multiplicative generalized coho-} \\ \text{mology theory} \end{array} \right\}.$$

Then we have $\mathrm{cup}(g, f; h) \le \mathrm{cup}(g, f) \le \mathrm{cat}(g, f)$ for any multiplicative generalized cohomology h. When h is the ordinary cohomology with a coefficient ring R, we denote $\mathrm{cup}(g, f; h)$ by $\mathrm{cup}(g, f; R)$. This definition immediately implies the following.

Remark 7.2. *For $(g, f) = (X; K, L; A)$, using the arguments in [14], we have*
$$\mathrm{cup}(g, f) = \mathrm{Min}\left\{ m \ge 0 \,\middle|\, \tilde{\Delta}_K^{m+1} : K/A \to X/L \wedge \textstyle\bigwedge^m X/A \text{ is stably trivial} \right\}.$$

Let us recall that Rudyak [17] and Strom [20] introduced a homotopy theoretical version of Fadell-Husseini's category weight (see [5]). But unfortunately, we could not succeed to give a version of category weight for our new relative L-S category. In this paper, we give instead a version of module weight which is a better computable lower estimate for our relative L-S category than cup length: let $f : (L, A) \subset (X, A)$ and $g : (K, A) \to (X, A)$ be maps in $\mathcal{T}^A$ and let h be a generalized cohomology theory.

Definition 7.3 (I. [14])**.** *A homomorphism $\phi : h^*(Y, L) \to h^*(K, A)$ of h_*-modules is called a (unstable) h-morphism if it preserves the action of any (unstable) cohomology operation on h^*.*

Definition 7.4. *A (unstable) module weight* $\mathrm{Mwgt}(g, f; h)$ *of (g, f) with respect to h is defined as follows.*

$$\mathrm{Mwgt}(g, f; h) = \mathrm{Min}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \text{There is a (unstable) } h\text{-morphism } \phi : \\ h^*(P^m(\Omega(X, L)), L) \to h^*(K, A) \text{ such that} \\ \phi\circ(e^X_m)^* = g^* : h^*(X, L) \to h^*(K, A). \end{array} \right\}$$

When h is the ordinary cohomology theory with coefficients in a ring R, we denote $\mathrm{Mwgt}(g, f; h)$ by $\mathrm{Mwgt}(g, f; R)$.

Remark 7.5. *The invariants introduced in this paper satisfy the following inequality for any generalised cohomology theory h^*:*

$$\mathrm{cup}(g, f; h) \leq \mathrm{Mwgt}(g, f; h) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f),$$

and hence for any ring R, we have

$$\mathrm{cup}(g, f; R) \leq \mathrm{Mwgt}(g, f; R) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f).$$

Similar to the above definition of $\mathrm{cup}(g, f)$, we define the following invariants.

Definition 7.6. *For any $(g, f) = (X; K, L; A)$, we define*

$$\mathrm{Mwgt}(g, f) = \mathrm{Max}\left\{ m{\geq}0 \;\middle|\; \begin{array}{l} \mathrm{Mwgt}(g, f; h) = m \text{ for some generalized cohomol-} \\ \text{ogy theory } h \end{array} \right\}$$

Remark 7.7. $\mathrm{cup}(g, f) \leq \mathrm{Mwgt}(g, f) \leq \mathrm{cat}(g, f) = \mathrm{catlen}(g, f)$.

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