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Co-H-spaces and the Ganea conjecture

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Abstract

A non-simply connected co-H-space X is, up to homotopy, the total space of a fibrewise-simply connected pointed fibrewise co-Hopf fibrant $j : X \to B\pi_1(X)$, which is a space with a co-action of $B\pi_1(X)$ along j. We construct its homology decomposition, which yields a simple construction of its fibrewise localisation. Our main result is the construction of a series of co-H-spaces, each of which cannot be split into a one-point-sum of a simply connected space and a bunch of circles, thus disproving the Ganea conjecture.

Problem 10 posed by Tudor Ganea [8], known as the Ganea conjecture (e.g, §6 in Arkowitz [1]), states: Does a co-H-space have the homotopy type of a one-point-sum of *a bunch of circles* (one-point-sum of S^1's or a point) and a simply connected space?

If a CW complex X is a co-H-space, the co-H-structure gives a co-action (see Berstein and Dror [3] or Oda [16]) of the classifying space $B\pi_1(X)$ of $\pi_1(X)$ along $j : X \to B\pi_1(X)$, the classifying map of the universal covering $p(X) : \widetilde{X} \to X$. It is known by Eilenberg-Ganea [6] or [11], that $\pi_1(X)$ is free and $B\pi_1(X)$ has the homotopy type of a bunch of circles, say B. Let $i : B \to X$ be a map representing a collection of generators of the free group $\pi_1(X)$ and $c : X \to C = X/B$ be the collapsing map from X to its cofibre. Clearly, we may choose the map i so that $j{\circ}i \sim 1_X$. It is also known by Corollary 3.4 and Theorem 3.3 in [11] that, for a given μ, a co-H-structure for X, there is a 'natural' map $s = s(\mu) : C = X/B \to X$ which is a right homotopy inverse of c. More precisely, if $f : (X, \mu) \to (X', \mu')$ is a co-H-map, then $f{\circ}s(\mu) = s(\mu'){\circ}f'$, where $f' : X/B \to X'/B'$ is the unique map induced from f. Hence

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one obtains two 'natural' homology equivalences $X \to B{\vee}C$ and $B{\vee}C \to X$, both of which induce isomorphisms of fundamental groups. As is well-known, these properties, however, do *not* guarantee that the two spaces have the same homotopy type.

Definition. A co-H-space is *standard* if it splits into a one-point-sum of a simply connected co-H-space and a bunch of circles.

Berstein and Dror [3] showed that a co-H-space is standard if the associated co-action is co-associative. Hilton, Mislin and Roitberg [10] showed that a co-H-space is standard if $e = i{\circ}j$ is 'loop-like' in $[X, X]$. We summarise here the relevant results of [3], [10].

Theorem 0.1 *([3], [10]) For a co-H-space complex X, the condition 1) below is equivalent with the conditions 2) to 5) below by several authors.*

1) *(Ganea [8]) A co-H-space is standard.*

2) *(Berstein-Dror [3]) The co-action of B along $j : X \to B$ associated with the co-H-structure of X can be chosen as co-associative.*

3) *(Hilton-Mislin-Roitberg [10]) The co-H-structure of X can be chosen to make the co-shear map a homotopy equivalence.*

4) *(Hilton-Mislin-Roitberg [10]) The co-H-structure of X can be chosen to be a co-loop, i.e, there is a natural algebraic-loop structure on the homotopy set functor $[X, -]$.*

5) *(Hilton-Mislin-Roitberg [10]) The co-H-structure of X can be chosen to make $e = i{\circ}j$ loop-like from the left (or right).*

However, we don't know any algorithm to get a nice co-H-structure from a given one.

On the other hand, there are some results on the conjecture which are shown without making any assumption on the co-H-structure itself: In [9], Henn verified the *almost rational* version of the conjecture:

Theorem 0.2 *([9]) An almost rational co-H-space is standard. Moreover it can be split into a one-point-sum of a rational spheres with dimensions ≥ 2 and a bunch of circles.*

In [14], Komatsu verified the conjecture for co-H-spaces with reduced homology groups free abelian and concentrated in one dimension other than 1. In [11], the Ganea conjecture is verified for co-H-spaces up to dimension 3:

Theorem 0.3 *([14], [11]) A co-H-space X is standard if the reduced homology group $\bar{H}_q(X)$ is trivial unless $q = 1$, $n+1$ or $n+2$, with $H_{n+2}(X)$ torsion free, for some $n \geq 1$.*

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1 Results

From now on, we work in the category of spaces having the homotopy type of a path-connected CW complex of finite type. The triple ($j : X \to B$, F, $i : B \to X$) stands for a pointed *fibrant* (see James [13] and [12], while the notion goes back to Quillen [17]), i.e. j is a fibration with fibre F and i is a closed cross-section of j. In the category of a pointed fibrants, there are (categorical) coproducts and products: For pointed fibrants (j_1,F_1,i_1) and (j_2,F_2,i_2), the former, denoted by $X_1{\vee_B}X_2$, is the push-out of the folding map $\nabla_B : B \to B{\vee}B$ and the section map $i_1{\vee}i_2$, and the latter, denoted by $X_1{\times_B}X_2$, is the pull-back of the diagonal map $\Delta_B : B \to B{\times}B$ and the fibration $j_1{\times}j_2$.

We assume that a pointed fibrant (j,F,i) is fibrewise-simply connected, i.e. F is simply connected. Then j and i induce maps $\widetilde{j} : \widetilde{X} \to \widetilde{B}$ and $\widetilde{i} : \widetilde{B} \to \widetilde{X}$ of universal coverings, and we have another pointed fibrant ($\widetilde{j}$, F, $\widetilde{i}$). We consider the following property:

$$\begin{array}{ccc} H_*(\widetilde{X},\widetilde{B}) & \xrightarrow{\cong} & \mathbb{Z}\pi{\otimes}H_*(X,B) \\ {\scriptstyle p(X)_*}\downarrow & \text{commutative} & \downarrow{\scriptstyle \mathbb{Z}\otimes_{\mathbb{Z}\pi}(-)} \\ H_*(X,B) & = & H_*(X,B), \end{array} \tag{1.1}$$

where $\pi = \pi_1(X)$. By [11] and Fox [7], we have the following result.

Theorem 1.1 *A co-H-space is, up to homotopy, a fibrewise co-H-space over $B\pi$ satisfying the above property (1.1).*

Proof. We may assume that a co-H-space X is, up to homotopy, the total space of a fibration $j : X \to B = B\pi$ the classifying map of $p(X) : \widetilde{X} \to X$. Then by [11], j satisfies (1.1) and the natural map $\hat{p}(X) : B{\vee}\widetilde{X} \to X$ (given by $\hat{p}(X)|_B = i$ and $\hat{p}(X)|_{\widetilde{X}} = p(X)$) has a homotopy section s. Let us recall that the universal covering of a co-H-space is also a co-H-space, since the Lusternik-Schnirelmann category of $\widetilde{X}$ cannot exceed that of X by Fox [7]. Hence there is a co-H-structure $\widetilde{\mu}$ on $\widetilde{X}$. By the definition of limits and colimits in the category of pointed fibrants, we know that $\hat{p}(X_1){\vee_B}\hat{p}(X_2) : (B{\vee}\widetilde{X_1}){\vee_B}(B{\vee}\widetilde{X_2}) = B{\vee}\widetilde{X_1}{\vee}\widetilde{X_2} \to X_1{\vee_B}X_2$ is given by $\hat{p}(X_1){\vee_B}\hat{p}(X_2)|_{B\vee\widetilde{X_t}} = \hat{p}(X_t)$, for $t = 1, 2$. By putting $\mu_B = (\hat{p}(X){\vee_B}\hat{p}(X)){\circ}(1_B{\vee}\widetilde{\mu}){\circ}s$, we get a fibrewise co-H-structure on $j : X \to B$. *qed.*

It is known that a simply connected CW complex has a Cartan-Serre-Whitehead decomposition, or a homology decomposition (see [5]). The property (1.1) yields the following result.

Theorem 1.2 *If j satisfies (1.1), then there exists a sequence of fibrewise-simply connected pointed fibrants ($j_n : X_n \to B$, F_n, $i_n : B \to X_n$) satisfying (1.1) with $X_1 = B$, $F_1 = \{*\}$ and $j_1 = 1_B = i_1$, which satisfies the following conditions for each $n \geq 1$:*

1) $l_n : X_n \hookrightarrow X_{n+1}$ and $X_n \hookrightarrow \bigcup_m X_m \simeq X$ are maps of pointed fibrants.

2) There is a map $h_n : S_n \xrightarrow{h'_n} F_n \subset X_n$, where S_n denotes the Moore space of type ($H_{n+1}(X,B)$,n) such that $S_n \xrightarrow{h_n} X_n \xhookrightarrow{l_n} X_{n+1}$ is a cofibre sequence up to homotopy.

3) The inclusion $m_n : X_n \subset X$ induces an isomorphism of fundamental groups.

4) The inclusion $m_n : X_n \subset X$ induces an isomorphism of homology groups of the universal coverings in dimensions $\leq n$ and $H_q(\widetilde{X}_n, \widetilde{B}) = 0$ for $q > n$.

Remark. The properties imply $h_1 \sim 0$, $X_2 \simeq B \vee \Sigma S_1$ and $X_{n+1} \simeq C(h_n)$, the cofibre of h_n. We call this an *almost homology decomposition* for a fibrewise-simply connected and pointed fibrant satisfying (1.1). For the k'-invariants of a co-H-space, we can show the following results.

Theorem 1.3 *If $(j,\ F,\ i)$ admits a fibrewise co-H-structure satisfying (1.1), then there are induced fibrewise co-H-structures on $(j_n,\ F_n,\ i_n)$ such that the inclusions $l_n : X_n \hookrightarrow X_{n+1}$ and $m_{n+1} : X_{n+1} \hookrightarrow X$ are fibrewise co-H-maps and the k'-invariants h_n are of finite order, $n \geq 1$.*

Corollary 1.3.1 *If X is a co-H-space, then each k'-invariant h_n is of finite order, $n \geq 1$.*

A fibrewise localisation and a fibrewise completion of a pointed fibrant is constructed by May [15]. If we make the additional assumption (1.1), there is a much simpler construction of fibrewise localisation using Theorem 1.2:

Theorem 1.4 *Let $\mathbb{P}$ be a set of primes. If j is a fibrewise-simply connected pointed fibrant satisfying (1.1), there is a fibrewise $\mathbb{P}$-localisation $\ell^B_{\mathbb{P}} : X \to X^B_{\mathbb{P}}$ which induces an isomorphism of fundamental groups and a homomorphism between reduced homology groups of the fibres which is given by tensoring with $\mathbb{Z}_{\mathbb{P}}$, the ring of $\mathbb{P}$-local integers.*

When $B \simeq B\pi_1(X)$, a fibrewise $\mathbb{P}$-localisation was constructed by Bendersky [2]. In that case, we will refer to a fibrewise localisation as an *almost localisation*.
Remark. By Theorem 1.4 and Corollary 1.3.1, we obtain another proof of Theorem 0.2.

By using the arguments given in [11], we obtain the following result (see Sections 5-8):

Theorem 1.5 *There is a series of co-H-spaces R_n, $n \geq 4$, with reduced homology groups free abelian and concentrated in dimensions 1, $n+1$ and $n+5$, such that each R_n is not standard.*

We say that a co-H-space X is of *stable dimension* k if its reduced homology $\bar{H}_q(X)$ is trivial unless $q = 1$ or $n+1 \leq q \leq n+k$, with $\bar{H}_{n+k}(X)$ torsion free, for some $n \geq 1$. We still don't know about the Ganea conjecture for a co-H-space of stable dimensions 3 and 4.

In the localised homotopy category, we have been unable to construct any counter examples to the conjecture. So we may state here the following local version of the Ganea conjecture:

Conjecture 1.6 *The almost p-localisation of a co-H-space is standard, for any prime p.*

Using the arguments given in Section 8, one can show that the non-trivial k'-invariants of the spaces in Theorem 1.5 are co-H-maps with respect to some non-standard co-H-structures.

2 Homology decomposition

In this section, we prove Theorem 1.2. Let S_n be the Moore space of type $(H_{n+1}(X,B),n)$, $n \geq 1$. For the first step, since $H_2(\widetilde{X},\widetilde{B}) \cong \mathbb{Z}\pi \otimes H_2(X,B)$ by [11], we have

$$\pi_2(F) \cong \pi_2(X,B) \cong \pi_2(\widetilde{X},\widetilde{B}) \cong H_2(\widetilde{X},\widetilde{B}) \cong \mathbb{Z}\pi \otimes H_2(X,B) \supset H_2(X,B). \tag{2.1}$$

Hence there exists a map $f_2 : \Sigma S_1 \to F \subset X$ representing a complete collection of generators of the $\mathbb{Z}\pi$-module $\pi_2(F)$ corresponding to (2.1). We deform the first projection $j_2' : X_2' = B \vee \Sigma S_1 \to B$ to a fibration up to homotopy, say $j_2 : X_2 \to B$, with fibre F_2, which satisfies (1.1) by (2.1). We define $g_2 : X_2 \to X$ by $g_2|_B = i$ and $g_2|_{\Sigma S_1} = f_2$. We can easily check that g_2 induces an isomorphism of fundamental groups, an isomorphism $\widetilde{g}_{2*} : \tilde{H}_q(\widetilde{X}_2) \to \tilde{H}_q(\widetilde{X})$ for $q \le 2$ and $\tilde{H}_q(\widetilde{X}_2) = 0$ for $q > 2$. We will consider g_2 as an inclusion.

We proceed to the next step: By (1.1), we have

$$\pi_3(F, F_2) \cong \pi_3(X, X_2) \cong H_3(\widetilde{X}, \widetilde{X_2}) \cong H_3(\widetilde{X}, \widetilde{B}) \cong \mathbb{Z}\pi \otimes H_3(X, B) \supset H_3(X, B). \tag{2.2}$$

Hence there exists a map $f_3 : (C(S_2), S_2) \to (F, F_2) \subset (X, X_2)$ representing a complete collection of generators of the $\mathbb{Z}\pi$-module $\pi_3(F, F_2)$ corresponding to (2.2). We put $h_2 = f_3|_{S_2}$ and deform the projection $j_3' : X_3' = X_2 \cup_{h_2} C(S_2) \to B \vee \Sigma S_2 \overset{\mathrm{pr}_B}{\to} B$ to a fibration up to homotopy, say $j_3 : X_3 \to B$ with fibre F_3, which satisfies (1.1) by (2.2). We define $g_3 : X_3 \to X$ by $g_3|_{X_2} = g_2$ and $g_3|_{C(S_2)} = f_3$. One can easily check that g_3 induces an isomorphism of fundamental groups, an isomorphism $\widetilde{g}_{3*} : \tilde{H}_q(\widetilde{X}_3) \to \tilde{H}_q(\widetilde{X})$ for $q \le 3$ and $\tilde{H}_q(\widetilde{X}_3, \widetilde{B}) = 0$ for $q > 3$. We will consider g_3 as an inclusion.

One can continue this process and get the fibrewise homology decomposition satisfying (1.1), for a finite complex. By using the telescope construction on the X_i's, we can also get the fibrewise homology decomposition satisfying (1.1), for an infinite complex. This completes the proof of Theorem 1.2.

3 Fibrewise localisation

In this section we prove Theorem 1.4. By Theorem 1.2, we have the homology decomposition $\{(j_n, F_n, i_n; h_n)\}_{n\ge1}$. We define the fibrewise $\mathbb{P}$-localisation $\ell^B_{\mathbb{P}} : j_n \to j_{n\mathbb{P}}^{\ B}$ by performing a step-by-step construction: Firstly, we know that $X_2 \simeq B \vee \Sigma S_1$. So we define $j_{2\mathbb{P}}^{\ B} : X_{2\mathbb{P}}^{\ B} \to B$ by deforming the first projection $\mathrm{pr}_B : B \vee (\Sigma S_1)_{\mathbb{P}} \to B$ into a fibrant and $\ell^B_{\mathbb{P}} : X_2 \to X_{2\mathbb{P}}^{\ B}$ by deforming $1_B \vee \ell_{\mathbb{P}} : X_2 = B \vee \Sigma S_1 \to B \vee (\Sigma S_1)_{\mathbb{P}}$ into a fibrewise map. Let F_2' be the fibre of $j_{2\mathbb{P}}^{\ B}$ which is homotopy equivalent to the fibre of $\widetilde{j_{2\mathbb{P}}^{\ B}} : \widetilde{X_{2\mathbb{P}}^{\ B}} \to \widetilde{B}$. Then by the Serre spectral sequence for $\widetilde{j_{2\mathbb{P}}^{\ B}}$, we have that the homology of F_2' is $\mathbb{P}$-local. Since F_2' is simply connected, F_2' itself is $\mathbb{P}$-local and can be regarded as the $\mathbb{P}$-localisation $F_{2\mathbb{P}}$ of F_2.

Secondly, let us recall that $X_3 \simeq X_3' = X_2 \cup_{h_2} C(S_2)$ and consider the following diagram:

$$
\begin{tikzcd}
 & F_2 \arrow[dr, hook] \arrow[ddddd, "\ell_{\mathbb{P}}"] & & & & & & \\
S_2 \arrow[ur, "h_2'"] \arrow[rr, "h_2"] \arrow[dddd, "\ell_{\mathbb{P}}"'] & & X_2 \arrow[rr, hook, "l_2'"] \arrow[dr, "j_2"] \arrow[ddd, "\ell_{\mathbb{P}}^B"'] & & X_3' \arrow[rr, "\simeq"] \arrow[dr, "j_3'"] \arrow[ddd, dashed, "\ell'^B_{\mathbb{P}}"'] & & X_3 \arrow[dr, "j_3"] \arrow[ddd, dashed, "\ell^B_{\mathbb{P}}"'] & \\
 & & & B \arrow[rr, equal] \arrow[d, equal] & & B \arrow[rr, equal] \arrow[d, equal] & & B \arrow[d, equal] \\
 & & & B \arrow[rr, equal] & & B \arrow[rr, equal] & & B \\
S_{2\mathbb{P}} \arrow[rr, dashed, "h_{2\mathbb{P}}^B"] \arrow[dr, dashed, "h_{2\mathbb{P}}'"'] & & X_{2\mathbb{P}}^B \arrow[ur, "j_{2\mathbb{P}}^B"] \arrow[rr, hook, dashed, "l'^B_{2\mathbb{P}}"'] & & X'^B_{3\mathbb{P}} \arrow[ur, dashed, "j'^B_{3\mathbb{P}}"] \arrow[rr, dashed, "\simeq"'] & & X^B_{3\mathbb{P}}. \arrow[ur, dashed, "j^B_{3\mathbb{P}}"] & \\
 & F_{2\mathbb{P}} \arrow[ur, hook] & & & & & &
\end{tikzcd}
\tag{3.1}
$$

By the universality of $\mathbb{P}$-localisation $\ell_{\mathbb{P}}$, $\ell_{\mathbb{P}}\circ h_2'$ induces the dotted arrow $h_{2\mathbb{P}}'$ such that $\ell_{\mathbb{P}}\circ h_2' \sim h_{2\mathbb{P}}'\circ\ell_{\mathbb{P}}$. Thus we can define $h_{2\mathbb{P}}^B$ as the composition map : $S_{2\mathbb{P}} \overset{h_{2\mathbb{P}}'}{\to} F_{2\mathbb{P}} \subset X_{2\mathbb{P}}^B$ and $X_{2\mathbb{P}}^B \overset{l'^B_{2\mathbb{P}}}{\hookrightarrow} X'^B_{3\mathbb{P}} = X_{2\mathbb{P}}^B \cup_{h_{2\mathbb{P}}^B} C(S_{2\mathbb{P}})$ as its cofibre. Since the image of $h_{2\mathbb{P}}^B$ lies in the fibre of $j_{2\mathbb{P}}^B$, the composition $j_{2\mathbb{P}}^B\circ h_{2\mathbb{P}}^B$ is trivial, and hence we can extend $j_{2\mathbb{P}}^B$ to the projection $j'^B_{3\mathbb{P}} : X'^B_{3\mathbb{P}} = X_{2\mathbb{P}}^B \cup_{h_{2\mathbb{P}}^B} C(S_{2\mathbb{P}}) \to B\vee\Sigma S_{2\mathbb{P}} \overset{\mathrm{pr}_B}{\to} B$ so that $j'^B_{3\mathbb{P}}\circ l'^B_{2\mathbb{P}} = j_{2\mathbb{P}}^B$ and $j'^B_{3\mathbb{P}}\circ\ell'^B_{\mathbb{P}} = j_3'$. So we define $j_{3\mathbb{P}}^B : X_{3\mathbb{P}}^B \to B$ by deforming $j'^B_{3\mathbb{P}} : X'^B_{3\mathbb{P}} \to B$ into a fibrant and $l_{2\mathbb{P}}^B : X_{2\mathbb{P}}^B \hookrightarrow X_{3\mathbb{P}}^B$ by deforming $l'^B_{2\mathbb{P}}$ into a fibrewise map. Then we remark that all the dotted arrows in the diagram (3.1) can be solidified so as to create a commutative diagram.

By continuing this process, we get the fibrewise $\mathbb{P}$-localisation $\ell_{\mathbb{P}}^B : X \to X_{\mathbb{P}}^B$ for a finite complex X. By using the telescope construction, we can also get the fibrewise $\mathbb{P}$-localisation $\ell_{\mathbb{P}}^B : X \to X_{\mathbb{P}}^B$ for an infinite complex X. This completes the proof of Theorem 1.4.

4 Homology decomposition of a co-H-space over B

In this section, we prove Theorem 1.3. Let $\mu : X \to X\vee_B X$ be any given fibrewise co-H-structure for j. We show the existence of the desired fibrewise co-H-structure μ_{n+1} for j_{n+1} by induction on $n \geq 0$. Since $X_1 = B$, μ induces the trivial fibrewise co-H-structure $\mu_1 = 1_B$ on $j_1 = 1_B$, which is clearly the restriction of μ to $X_1 = B$.

Let $n \geq 1$. Firstly we prove that j_{n+1} admits a fibrewise co-H-structure μ''_{n+1}: When $n = 1$, since $X_2 \simeq B \vee \Sigma S_1$, there is a standard fibrewise co-H-structure μ''_2 on j_2 as an extension of the trivial co-H-structure μ_1, that is $(l_1\vee_B l_1)\circ\mu_1 \sim \mu''_2\circ l_1$. Thus we may assume that $n \geq 2$. Then by the induction hypothesis, there is a fibrewise co-H-structure μ_n on j_n which is a compression of $\mu|_{X_n}$. Then the map $\gamma = (l_n\vee_B l_n)\circ\mu_n\circ h_n : S_n \to X_{n+1}\vee_B X_{n+1}$ gives the obstruction to extend $(l_n\vee_B l_n)\circ\mu_n$ on X_{n+1}. We regard $\gamma \in \pi_n(X_{n+1}\vee_B X_{n+1}; G)$, $G = H_{n+1}(X, B)$. By the induction hypothesis, we have $(m_{n+1}\vee_B m_{n+1})\circ\gamma = (m_n\vee_B m_n)\circ\mu_n\circ h_n \sim \mu\circ m_n\circ h_n = \mu\circ m_{n+1}\circ l_n\circ h_n \sim 0.$

Hence there is an element $\hat{\gamma} \in \pi_{n+1}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; G)$ such that $\partial(\hat{\gamma}) = \gamma$ in the following commutative diagram with exact rows:

$$\begin{CD}
\pi_{n+1}(X \vee_B X; G) @>{l'_*}>> \pi_{n+1}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; G) @>{\partial}>> \pi_n(X_{n+1} \vee_B X_{n+1}; G) \\
@V{k_*}VV @V{k'_*}VV @V{k_{n+1*}}VV \\
\pi_{n+1}(X \times_B X; G) @>{l_*}>> \pi_{n+1}(X \times_B X, X_{n+1} \times_B X_{n+1}; G) @>{\partial}>> \pi_n(X_{n+1} \times_B X_{n+1}; G);
\end{CD} \tag{4.1}$$

here $k : X \vee_B X \hookrightarrow X \times_B X$, $k' : (X \vee_B X, X_{n+1} \vee_B X_{n+1}) \hookrightarrow (X \times_B X, X_{n+1} \times_B X_{n+1})$, $k_{n+1} : X_{n+1} \vee_B X_{n+1} \hookrightarrow X_{n+1} \times_B X_{n+1}$, $l' : X \vee_B X \hookrightarrow (X \vee_B X, X_{n+1} \vee_B X_{n+1})$ and $l : X \times_B X \hookrightarrow (X \times_B X, X_{n+1} \times_B X_{n+1})$ are the canonical inclusions.

To proceed, we show that k_* is a split epimorphism and k'_* is an isomorphism: Let us recall the Universal Coefficient Theorem due to Eckmann and Hilton: For any topological pair (U,V) and an abelian group G, there is the following short exact sequence for $q \geq 2$.

$$0 \to \mathrm{Ext}\,(G, \pi_{q+2}(U,V)) \to \pi_{q+1}(U,V;G) \to \mathrm{Hom}\,(G, \pi_{q+1}(U,V)) \to 0.$$

Applying this to the $n+1$-connected pair $(X \vee_B X, X_{n+1} \vee_B X_{n+1})$, using the Hurewicz isomorphism theorem and (1.1) for $n \geq 2$, we obtain

$$\begin{aligned}
\pi_{n+1}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; G) &\cong \mathrm{Ext}\,(G, H_{n+2}(\widetilde{X \vee_B X}, \widetilde{X_{n+1} \vee_B X_{n+1}}) \\
&\cong H_{n+2}(\widetilde{X \vee_B X}, \widetilde{X_{n+1} \vee_B X_{n+1}}; \mathrm{tor}\, G) \\
&\cong \mathbb{Z}\pi \otimes H_{n+2}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; \mathrm{tor}\, G),
\end{aligned}$$

Similarly for $n \geq 2$, we obtain

$$\begin{aligned}
\pi_{n+1}(X \times_B X, X_{n+1} \times_B X_{n+1}; G) &\cong \mathbb{Z}\pi \otimes H_{n+2}(X \times_B X, X_{n+1} \times_B X_{n+1}; \mathrm{tor}\, G) \\
&\cong \mathbb{Z}\pi \otimes H_{n+2}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; \mathrm{tor}\, G).
\end{aligned}$$

Thus $k'_* : \pi_{n+1}(X \vee_B X, X_{n+1} \vee_B X_{n+1}; G) \to \pi_{n+1}(X \times_B X, X_{n+1} \times_B X_{n+1}; G)$ is an isomorphism, $n \geq 2$. The pointed fibrewise space $X \vee_B X \to B$ has the fibre $F \vee F$, and hence $\pi_{n+1}(X \vee_B X; G)$ is isomorphic with $\pi_{n+1}(F \vee F; G) \oplus \pi_{n+1}(B)$, $n \geq 2$. The pointed fibrewise space $X \times_B X \to B$ has the fibre $F \times F$, and hence $\pi_{n+1}(X \times_B X; G)$ is isomorphic with $\pi_{n+1}(F \times F; G) \oplus \pi_{n+1}(B)$, $n \geq 2$. Since the homomorphism $\pi_{n+1}(F \vee F; G) \to \pi_{n+1}(F \times F; G)$ has a natural splitting $\sigma^F_* : \pi_{n+1}(F \times F; G) \to \pi_{n+1}(F \vee F; G)$, so does the homomorphism $k_* : \pi_{n+1}(X \vee_B X; G) \to \pi_{n+1}(X \times_B X; G)$ admit a natural splitting $\sigma^j_* : \pi_{n+1}(X \times_B X; G) \to \pi_{n+1}(X \vee_B X; G)$ with respect to j, $n \geq 2$.

On the other hand, since $k_n \circ \mu_n$ is homotopic to Δ_n, the fibrewise diagonal map in $X_n \times_B X_n$, we have $k_{n+1} \circ (l_n \vee_B l_n) \circ \mu_n = (l_n \times_B l_n) \circ k_n \circ \mu_n \sim (l_n \times_B l_n) \circ \Delta_n = \Delta_{n+1} \circ l_n$, and hence $k_{n+1} \circ \gamma \sim \Delta_{n+1} \circ l_n \circ h_n \sim 0$. Thus $\partial \circ k'_*(\hat{\gamma}) = k_{n+1*} \circ \partial(\hat{\gamma}) = k_{n+1*}(\gamma) = 0$, and hence there is an element $\gamma' \in \pi_{n+1}(X \times_B X; G)$ such that $l_*(\gamma') = k_*(\hat{\gamma})$. Since the left vertical arrow k_* is an epimorphism, γ' can be pulled back to an element $\gamma_0 \in \pi_{n+1}(X \times_B X; G)$. Hence $k'_* \circ l'_*(\gamma_0) = l_* \circ k_*(\gamma_0) = l_*(\gamma') = k_*(\hat{\gamma})$. Since k'_* is an isomorphism, we have that $\hat{\gamma} = l'_*(\gamma_0)$, and hence we get $\gamma =$

$\partial(\hat{\gamma}) = \partial\circ l'_*(\gamma_0) = 0$. Thus there is a map $\mu'_{n+1} : X_{n+1} \to X_{n+1}\vee_B X_{n+1}$ which is an extension of $(l_n\vee_B l_n)\circ\mu_n$.

Since X_{n+1} is, up to homotopy, the cofibre of $h_n : S_n \to F_n \subset X_n$, it admits a co-action of ΣS_n. Thus the "difference" between $k_{n+1}\circ\mu'_{n+1}$ and Δ_{n+1}, is given by a map $\delta : \Sigma S_n = X_{n+1}/X_n \to X_{n+1}\times_B X_{n+1}$ which can be pulled back to a map $\delta_0 : \Sigma S_n \to X_{n+1}\vee_B X_{n+1}$, since k_{n+1*} is an epimorphism. By "adding" δ_0 to μ'_{n+1}, we get μ''_{n+1}, a fibrewise co-H-structure on j_{n+1} as an extension of μ_n, that is $(l_n\vee_B l_n)\circ\mu_n \sim \mu''_{n+1}\circ l_n$.

Secondly, we prove the existence of a fibrewise co-H-structure μ_{n+1} such that $(l_n\vee_B l_n)\circ\mu_n \sim \mu_{n+1}\circ l_n$ and $(m_{n+1}\vee_B m_{n+1})\circ\mu_{n+1} \sim \mu\circ m_{n+1}$: Since $(m_{n+1}\vee_B m_{n+1})\circ\mu''_{n+1}$ and $\mu\circ m_{n+1}$ coincide when restricted to X_n, the "difference" between them is given by a map $\varepsilon : \Sigma S_n \to X\vee_B X$. We regard $\varepsilon \in \pi_{n+1}(X\vee_B X; G)$, $G = H_{n+1}(X, B)$. Since μ''_{n+1} and μ are fibrewise co-H-structures for j_{n+1} and j, we have $k\circ(m_{n+1}\vee_B m_{n+1})\circ\mu''_{n+1} = (m_{n+1}\times_B m_{n+1})\circ k_{n+1}\circ\mu''_{n+1} \sim (m_{n+1}\times_B m_{n+1})\circ\Delta_{n+1} = \Delta\circ m_{n+1} \sim k\circ\mu\circ m_{n+1}$. Hence $k_*(\varepsilon) = 0$ and $k'_*\circ l'_*(\varepsilon) = l_*\circ k_*(\varepsilon) = 0$. Since k'_* is an isomorphism, we have $l'_*(\varepsilon) = 0$, and hence ε can be pulled back to an element $\varepsilon'_0 \in \pi_{n+1}(X_{n+1}\vee_B X_{n+1}; G)$. Let $\varepsilon_0 = \varepsilon'_0 - \sigma_*^{j_{n+1}}\circ k_{n+1*}(\varepsilon'_0) \in \pi_{n+1}(X_{n+1}\vee_B X_{n+1}; G)$, where $\sigma_*^{j_{n+1}}$ is the splitting for k_{n+1*}. Then $k_{n+1*}(\varepsilon_0) = k_{n+1*}(\varepsilon'_0) - k_{n+1*}\circ\sigma_*^{j_{n+1}}\circ k_{n+1*}(\varepsilon'_0) = 0$ and

$$\begin{aligned}(m_{n+1}\vee_B m_{n+1})_*(\varepsilon'_0) &= (m_{n+1}\vee_B m_{n+1})_*(\varepsilon_0) - (m_{n+1}\vee_B m_{n+1})_*\circ\sigma_*^{X_{n+1}}\circ k_{n+1*}(\varepsilon'_0)\\ &= \varepsilon - \sigma_*^{X}\circ(m_{n+1}\times_B m_{n+1})_*\circ k_{n+1*}(\varepsilon'_0) = \varepsilon - \sigma_*^{X}\circ k_{n+1*}\circ m_{n+1*}(\varepsilon'_0) = \varepsilon.\end{aligned}$$

Thus by adding ε_0 to μ''_{n+1}, we get another fibrewise co-H-structure μ_{n+1} over B of X_{n+1}. One can easily check that μ_{n+1} has the desired properties.

Finally, we prove that the k'-invariant h_n is of finite order: We observe that when X is a fibrewise co-H-space, then the fibre F of $j : X \to B$ is a simply connected genuine co-H-space. The k'-invariant $h_n : S_n \to F_n \subset X_n$ is the composition of the k'-invariant h'_n for the simply connected co-H-space F and the inclusion $F_n \hookrightarrow X_n$. Since $h'_n : S_n \to F_n$ is of finite order, by Theorem I in Curjel [4], h_n is also of finite order. This completes the proof of Theorem 1.3.

5 Construction of a complex R_n for $n \geq 4$

The remainder of this paper is devoted to proving Theorem 1.5. In this section, we construct the complex R_n: Let $A_n = S^{n+1}$ and $B = S^1$. We define C_n as the following complex:

$$C_n = S^{n+1} \cup_{\Sigma^{n-3}\alpha+\Sigma^{n-3}\beta} e^{n+5} = \Sigma^{n-3}C_4, \quad C_4 = S^4 \cup_{\nu_4} e^8 = \mathbb{H}P^2, \quad \alpha = 9\nu_4,\ \beta = -8\nu_4,$$

where $\nu_4 : S^7 \to S^4$ denotes the Hopf map. The complex R_n is defined as follows:

$$R_n = (B\vee A_n) \cup_{\mathrm{in}_{A_n}\circ\Sigma^{n-3}\alpha+\psi(\tau)\circ\mathrm{in}_{A_n}\circ\Sigma^{n-3}\beta} e^{n+5},$$

where in_{A_n} denotes the inclusion $A_n \hookrightarrow B\vee A_n$ and $\psi : \pi \to \pi_0\mathrm{Map}_*(B\vee A_n, B\vee A_n)$ denotes the action of the fundamental group $\pi = \langle\tau\rangle \cong \mathbb{Z}$ of $B\vee A_n$ on itself. We remark that the image of ψ is in the group of homotopy classes of self homotopy equivalences $\mathrm{Aut}\,(B\vee A_n)$. Let

$p^{R_n} : \widetilde{R_n} \to R_n$ be the universal covering of R_n. By the definition of R_n, the homotopy type of $\widetilde{R_n}$ is as follows:

$$\widetilde{R_n} \simeq (\bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n) \cup (\bigvee_{j\in\mathbb{Z}} \tau^j{\cdot}e^{n+5}) \quad \text{and} \quad \widetilde{B\vee A_n} = \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n,$$

where we denote by $\widetilde{\psi(\tau^i)} : \widetilde{B\vee A_n} \to \widetilde{B\vee A_n}$ the map induced from $\psi(\tau^i)$ on the universal coverings. Also $\tau^i{\cdot}(-)$ stands for $\widetilde{\psi(\tau^i)}(-)$. Here, the attaching map of the cell $1{\cdot}e^{n+5}$ is given by the suspension map

$$S^{n+4} \overset{\{\Sigma^{n-3}\alpha,\Sigma^{n-3}\beta\}}{\to} A_n\vee A_n \overset{1_{A_n}\vee\widetilde{\psi(\tau)}}{\to} A_n\vee\tau{\cdot}A_n \subset \bigvee_{i\in\mathbb{Z}} \tau^i{\cdot}A_n.$$

We define a projection $p : B\vee\widetilde{R_n} \to R_n$ by putting

$$p|_B : B \overset{\mathrm{in}_B}{\hookrightarrow} B\vee A_n \subset R_n, \quad \text{and} \quad p|_{\widetilde{R_n}} = p^{R_n} : \widetilde{R_n} \to R_n.$$

Let $p_0 = p|_{B\vee\bigvee_{i\in\mathbb{Z}}\tau^i\cdot A_n} : B\vee\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n \to B\vee A_n$. Then we have $p_0|_{\tau^j\cdot A_n} : \tau^j{\cdot}A_n \overset{\simeq}{\to} \psi(\tau^j)(A_n)$ $\subset B\vee A_n$, and hence, $p_0|_{\Psi(\tau^i)(\tau^j\cdot A_n)} : \Psi(\tau^i)(\tau^j{\cdot}A_n) = \Psi(\tau^i)(\widetilde{\psi(\tau^j)}(A_n)) \overset{\simeq}{\to} \psi(\tau^{i+j})(A_n) \subset B\vee A_n$, where Ψ denotes the action of π on $B\vee\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n$.

6 Self maps of $A_n = S^{n+1}$

This section provides an easy but rather crucial property of R_n for $n \geq 4$. Let $f : A_n \to A_n$ and $g : A_n \to A_n$ be maps of degrees -8 and 9. We obtain

$$f + g \sim 1_{A_n}. \tag{6.1}$$

We know the following equations modulo 24, the order of $\Sigma^{n-3}\nu_4 = \nu_{n+1}$:

$$(-8)^2 \equiv -8, \quad 9^2 \equiv 9, \quad (-8)\times 9 = 9\times(-8) \equiv 0 \mod 24.$$

Since $\Sigma^{n-3}\alpha = 9\nu_{n+1}$ and $\Sigma^{n-3}\beta = -8\nu_{n+1}$, these equations imply the following properties:

Proposition 6.1 *The compositions of f and g with $\Sigma^{n-3}\alpha$ and $\Sigma^{n-3}\beta$ give the equations:*
(1) $f{\circ}\Sigma^{n-3}\alpha \sim *$, (2) $g{\circ}\Sigma^{n-3}\alpha \sim \Sigma^{n-3}\alpha$, (3) $g{\circ}\Sigma^{n-3}\beta \sim *$ and (4) $f{\circ}\Sigma^{n-3}\beta \sim \Sigma^{n-3}\beta$.

7 Homotopy section of $B\vee\widetilde{R_n} \to R_n$

By Theorem 3.3 in [11], the existence of a homotopy section of $p : B\vee\widetilde{R_n} \to R_n$ is a necessary and sufficient condition for R_n to admit a co-action of B along $j : R_n \to B$. Here the universal covering $\widetilde{R_n}$ of R_n is desuspendable for dimensional reasons. Hence the existence of a homotopy section of p implies that R_n is a co-H-space. In summary:

Lemma 7.1 *The following two conditions on R_n are equivalent.*

1) There is a homotopy section of $p : R_n \to B{\vee}\widetilde{R_n}$.

2) R_n admits a co-H-structure.

Now we show the existence of a homotopy section of $p : B{\vee}\widetilde{R_n} \to R_n$. We define a map $s_0 : B{\vee}A_n \to B{\vee}\widetilde{B{\vee}A_n} \simeq B{\vee}\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n$ as follows:

$$s_0|_B = \mathrm{in}_B : B \to B{\vee}\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n,$$

$$s_0|_{A_n} : A_n \overset{\{f,g\}}{\to} A_n{\vee}A_n \overset{\widetilde{\psi(\tau)}\vee 1_{A_n}}{\longrightarrow} \tau{\cdot}A_n{\vee}A_n \overset{(\Psi(\tau^{-1})\circ\mathrm{in}_{\tau\cdot A_n})\vee 1_{A_n}}{\longrightarrow} B{\vee}\tau{\cdot}A_n{\vee}A_n \subset B{\vee}\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n.$$

By (6.1), we have $p_0{\circ}s_0 \sim 1_B{\vee}(f+g) \sim 1_B{\vee}1_{A_n} = 1_{B\vee A_n}$. Since $n \geq 4$, it follows that $\pi_{n+4}(A_n{\vee}A_n) \cong \pi_{n+4}(A_n){\oplus}\pi_{n+4}(A_n)$ for dimensional reasons. By Proposition 6.1, we have

$$s_0{\circ}\Sigma^{n-3}\alpha \sim \mathrm{in}_{A_n}{\circ}\Sigma^{n-3}\alpha : S^{n+4} \to B{\vee}A_n \subset B{\vee}\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n,$$

$$s_0{\circ}\Sigma^{n-3}\beta \sim \Psi(\tau^{-1}){\circ}\mathrm{in}_{\tau\cdot A_n}{\circ}\widetilde{\psi(\tau)}{\circ}\Sigma^{n-3}\beta : S^{n+4} \to A_n \to \tau{\cdot}A_n \to B{\vee}\tau{\cdot}A_n \subset B{\vee}\bigvee_{i\in\mathbb{Z}}\tau^i{\cdot}A_n.$$

Hence we obtain that

$$\begin{aligned} s_0{\circ}(\Sigma^{n-3}\alpha + \psi(\tau){\circ}\Sigma^{n-3}\beta) &= s_0{\circ}\Sigma^{n-3}\alpha + \Psi(\tau){\circ}s_0{\circ}\Sigma^{n-3}\beta \\ &\sim \mathrm{in}_{A_n}{\circ}\Sigma^{n-3}\alpha + \mathrm{in}_{\tau\cdot A_n}{\circ}\widetilde{\psi(\tau)}{\circ}\Sigma^{n-3}\beta = \mathrm{in}_{A_n\vee\tau\cdot A_n}{\circ}(\Sigma^{n-3}\alpha + \widetilde{\psi(\tau)}{\circ}\Sigma^{n-3}\beta). \end{aligned}$$

Thus the map $s_0{\circ}(\Sigma^{n-3}\alpha + \psi(\tau){\circ}\Sigma^{n-3}\beta)$ is homotopic to the attaching map of the cell $1{\cdot}e^{n+5}$. Hence it induces a map $s : R_n \to B{\vee}\widetilde{R_n}$ so that $p{\circ}s$ is clearly the identity up to homotopy.

By Lemma 7.1, we obtain the following theorem.

Theorem 7.2 *R_n is a co-H-space.*

8 Unsplittability of R_n

In this section, we show that R_n is not standard. We state the following well-known result:

Proposition 8.1 *The set of invertible elements in the group ring $\mathbb{Z}\pi$ is $\pm\pi \subset \mathbb{Z}\pi$.*

Proof. Since π is the infinite cyclic group, $\mathbb{Z}\pi$ is isomorphic with $\mathbb{Z}[x, \frac{1}{x}]$ the ring of Laurent polynomials with coefficients in $\mathbb{Z}$. We can express each Laurent polynomial in the form $x^i(a_\ell x^\ell + a_{\ell-1}x^{\ell-1} + ... + a_1x^1 + a_0)$ with $a_\ell a_0 \neq 0$, $\ell \geq 0$ and $i \in \mathbb{Z}$. If the product of any two such Laurent polynomials, say $x^i(a_\ell x^\ell + ... + a_0)$ and $x^j(b_m x^m + ... + b_0)$, is equal to the unity, then we have that $i + j = \ell = m = 0$ and $a_0b_0 = 1$. Hence every invertible element can be expressed as $\pm x^i$ for some $i \in \mathbb{Z}$. *qed.*

Let us assume that R_n has the homotopy type of a one-point-sum of a simply connected space C' and a bunch of circles B'. Since the fundamental group of R_n is clearly $\pi \cong \mathbb{Z}$, $B' = S^1 = B$ and the inclusion of B' in R_n is given by a generator $\tau^{\pm 1}$ of π. Since C' has the homotopy type of the mapping cone of the inclusion $B' \subset R_n$, $C' \simeq R_n/B = C_n$.

Thus our assumption implies that R_n has the homotopy type of $B \vee C_n$, which will lead us to a contradiction: Let $h : R_n \to B \vee C_n$ be a homotopy equivalence, which induces an isomorphism $\widetilde{h}_* : \tilde{H}_*(\widetilde{R_n};\mathbb{Z}) \to \tilde{H}_*(\widetilde{B \vee C_n};\mathbb{Z})$. As is easily seen, we have

$$\tilde{H}_*(\widetilde{R_n};\mathbb{Z}) \cong \mathbb{Z}\pi\{x_{n+1}, x_{n+5}\} \quad \text{and} \quad \tilde{H}_*(\widetilde{B \vee C_n};\mathbb{Z}) \cong \mathbb{Z}\pi\{u_{n+1}, u_{n+5}\}$$

where x_q and u_q are the homology classes corresponding to the q-cells in R_n and $B \vee C_n$, respectively. By Proposition 8.1, it follows that $\widetilde{h}_*(x_{n+1}) = \pm\tau^i u_{n+1}$ and $\widetilde{h}_*(x_{n+5}) = \pm\tau^j u_{n+5}$, for some $i, j \in \mathbb{Z}$. Using a suitable deck transformation on $\widetilde{B \vee C_n}$, we may assume that $i = 0$.

The (non-trivial) right actions of the Steenrod algebra on the homology groups $\tilde{H}_*(\widetilde{R_n};\mathbb{F}_p)$ and $\tilde{H}_*(\widetilde{B \vee C_n};\mathbb{F}_p)$ for $p = 2$ and $p = 3$ are given by the following proposition.

Proposition 8.2 *(1) Let x'_q be the modulo 2 reduction of the element x_q. Then, in $\tilde{H}_*(\widetilde{R_n};\mathbb{F}_2)$, the only non-trivial relation is: $x'_{n+5}\mathcal{S}q^4 = x'_{n+1}$,*

(2) Let u'_q be the modulo 2 reduction of the element u_q. Then, in $\tilde{H}_*(\widetilde{B \vee C_n};\mathbb{F}_2)$,the only non-trivial relation is: $u'_{n+5}\mathcal{S}q^4 = u'_{n+1}$.

(3) Let x''_q be the modulo 3 reduction of the element x_q. Then, in $\tilde{H}_*(\widetilde{R_n};\mathbb{F}_3)$, the only non-trivial relation is: $x''_{n+5}\mathcal{P}^1 = \tau\cdot x''_{n+1}$.

(4) Let u''_q be the modulo 3 reduction of the element u_q. Then, in $\tilde{H}_*(\widetilde{B \vee C_n};\mathbb{F}_3)$, the only non-trivial relation is: $u''_{n+5}\mathcal{P}^1 = u''_{n+1}$.

Thus in $\tilde{H}_{n+1}(\widetilde{B \vee C_n};\mathbb{F}_2)$ and $\tilde{H}_{n+1}(\widetilde{B \vee C_n};\mathbb{F}_3)$, we have the following equations:

$$u'_{n+1} = \widetilde{h}_*(x'_{n+1}) = \widetilde{h}_*(x'_{n+5}\mathcal{S}q^4) = \widetilde{h}_*(x'_{n+5})\mathcal{S}q^4 = \tau^j\cdot u'_{n+5}\mathcal{S}q^4 = \tau^j\cdot u'_{n+1},$$
$$u''_{n+1} = \pm\widetilde{h}_*(x''_{n+1}) = \pm\widetilde{h}_*(\tau^{-1}\cdot x''_{n+5}\mathcal{P}^1) = \pm\tau^{-1}\cdot\widetilde{h}_*(x''_{n+5})\mathcal{P}^1 = \pm\tau^{j-1}\cdot u''_{n+5}\mathcal{P}^1 = \pm\tau^{j-1}\cdot u''_{n+1}.$$

The upper line tells us that $j = 0$, while the lower line tells us that $j = 1$. This is a contradiction. Thus we obtain the following theorem.

Theorem 8.3 *R_n is not standard.*

Theorems 7.2 and 8.3 imply Theorem 1.5.

Remark. Although $R_n \not\simeq B \vee C_n$, we know that these spaces have isomorphic homotopy groups in each dimension, because their almost p-localisations are homotopy equivalent for any prime p. But we don't know whether the universal coverings of these spaces are homotopy equivalent or not, while the universal coverings are not $\pi_1(B)$-equivariant homotopy equivalent.

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