# ADJOINT ACTION OF A FINITE LOOP SPACE II

**NORIO IWASE**
GRADUATE SCHOOL OF MATHEMATICS,
KYUSHU UNIVERSITY ROPPONMATSU,
FUKUOKA 810, JAPAN
CURRENT: DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF ABERDEEN,
EDWARD WRIGHT BUILDING, DUNBAR STREET,
ABERDEEN AB24 3QY, SCOTLAND

**AKIRA KONO**
DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE, KYOTO UNIVERSITY,
KYOTO 606, JAPAN

Abstract. Adjoint actions of compact simply connected Lie groups are studied by Kozima and the second author based on the series of studies on the classification of simple Lie groups and their cohomologies. At odd primes, the first author showed that there is a homotopy theoretic approach that will prove the results of Kozima and the second author for any 1-connected finite loop spaces. In this paper, we use the rationalisation of the classifying space to compute the adjoint actions and the cohomology of classifying spaces assuming torsion free hypothesis, at the prime 2. And, by using Browder's work on the Kudo-Araki operations $Q_1$ for homotopy commutative Hopf spaces, we show the converse for general 1-connected finite loop spaces, at the prime 2. This can be done because the inclusion $j : G \to B\Lambda G$ satisfies the homotopy commutativity for any non-homotopy commutative loop space $G$.

## 1. Introduction

For a connected topological group $G$, the loop group $\Lambda G = \{u : S^1 \to G\}$ is homeomorphic with the product group $G \times \Omega G$, where $\Omega G$ denotes the subspace of loops start and end the unit $e \in G$. However the multiplication of $\Lambda G$ is different from that of the product group $G \times \Omega G$, unless $G$ is abelian. The difference can be described by the adjoint (left) action of $G$ on $\Omega G$, say $Ad : G \times \Omega G \to \Omega G$ by $Ad(g, \ell)(t) = g\ell(t)g^{-1}$. Kozima and the second author [?] studied the difference in terms of the cohomology of the classifying space, when $G$ is a 1-connected compact Lie group. In this paper, our approach is rather homotopy theoretical.

*Date*: 29th August 1997.
1991 *Mathematics Subject Classification.* Primary 55P45, 55Q25 Secondary 55N20.
*Key words and phrases.* loop group, loop space, 2-torsion.
This research was partially supported by Grant-in-Aid for Scientific Research (C)08640125 from The Ministry of Science, Sports and Culture.

The authors thank Katsuhiko Kuribayashi for valuable conversations and e-mails without which this work could not be completed and University of Aberdeen for its hospitality during the first author's stay in Aberdeen.

## 2. Main Theorems

First of all, let $G$ be a finite loop space, in other words, a topological group with the homotopy type of a finite CW complex.

Let us recall that the classifying space of $G\times\Omega G$ has the homotopy type of $\{BG\}\times G$, while the classifying space of $\Lambda G$ is given by $B\Lambda G = EG\times_G G$, where $EG$ denotes the total space of the universal principal $G$-bundle and the right and left actions of $G$ on $EG$ is the diagonal action and that on $G$ is the self adjoint (left) action $ad : G\times G \to G$ by $ad(g,h) = ghg^{-1}$. Thus there is the following fibration:

$$G \xrightarrow{j} B\Lambda G = EG\times_G G \xrightarrow{p} BG, \tag{2.1}$$

where the projection $p$ has a cross-section $s : BG \to EG\times_G G$, since the adjoint action leaves the unit fixed.

The following theorem is due to Hubbuck's Torus Theorem and we leave its proof to the reader.

**Theorem 2.1.** *The following three conditions are equivalent for $G$ a finite loop space.*

*a) The inclusion $j : G \to B\Lambda G$ has a homotopy left inverse: $B\Lambda G \to G$, at the prime 2.*

*b) There is a homotopy equivalence $\Phi : B\Lambda G \to BG\times G$, at the prime 2, which satisfies $\Phi j \simeq in_2$, where $in_2$ denotes the inclusion to the second factor.*

*c) $G$ has, at the prime 2, the homotopy type of a torus of some dimension, say $r \geq 0$, and then $BG$ has the homotopy type of the product of $r$ copies of $CP^\infty$.*

From now on we always assume that $G$ is a 1-connected finite loop space. The following result are obtained by assuming homological properties.

**Theorem 2.2.** *At the prime 2, the following three conditions are equivalent.*

*i) The integral homology $H_*(G;\mathbb{Z})$ has no 2-torsion.*

*ii) The adjoint action induces the trivial action $Ad_* = pr_{2*} : H_*(G;\mathbb{F}_2)\otimes H_*(\Omega G;\mathbb{F}_2) \to H_*(\Omega G;\mathbb{F}_2)$.*

*iii) There is an isomorphism of algebras $\Phi : H^*(B\Lambda G;\mathbb{F}_2) \cong H^*(\{BG\}\times G;\mathbb{F}_2)$ which satisfies $in_2^*\Phi = j^*$.*

By Theorem **??**, $B\Lambda G$ does not have the modulo 2 homotopy type of $\{BG\}\times G$ even when the conditions given in Theorem **??** is satisfied. In [**?**], Kozima and the second author proved the above theorem for 1-connected compact Lie groups. In this paper, for the prime 2, we only assume that $G$ is a topological group or a loop space with homotopy type of a finite complex. Our result needs a modification of Browder's work on the Kudo-Araki operations (see [**?**] and [**?**]). For the module structures, we also have the following result.

**Theorem 2.3.** *At the prime 2, the following two conditions are equivalent.*

*iv) The induced homomorphism $j^* : H^*(B\Lambda G;\mathbb{F}_2) \to H^*(G;\mathbb{F}_2)$ is surjective.*

*v) There is an $H^*(BG;\mathbb{F}_2)$-module isomorphism $\Phi : H^*(B\Lambda G;\mathbb{F}_2) \cong H^*(\{BG\}\times G;\mathbb{F}_2)$ which satisfies $in_2^*\Phi = j^*$.*

If we consider $ad_*$ instead of $Ad_*$, we only have the following result.

**Theorem 2.4.** *At the prime 2, the following four conditions are equivalent.*

*vi) The self adjoint action induces the trivial action $ad_* = pr_{2*} : H_*(G;\mathbb{F}_2)\otimes H_*(G;\mathbb{F}_2) \to H_*(G;\mathbb{F}_2)$.*

*vii) The Pontryagin ring $H_*(G;\mathbb{F}_2)$ is commutative Hopf algebra.*

*viii) The Hopf algebra $H^*(G;\mathbb{F}_2)$ is primitively generated.*

*ix) The Hopf algebra $H_*(G;\mathbb{F}_2)$ is an exterior algebra.*

We have the following relations among the conditions given in the above theorems.

**Theorem 2.5.** *The conditions given in Theorem* **??** *are stronger than those in Theorem* **??** *and the conditions given in Theorem* **??** *are stronger than those in Theorem* **??**.

**Example 2.6.** *It is well-known that*

*1) $SU(n)$'s and $Sp(n)$'s satisfy all the conditions in Theorems* **??**, **??** *and* **??**.

*2) $G_2$, $F_4$, $Spin(n)$ ($n \leq 9$) satisfy the conditions in Theorems* **??** *and* **??**, *but do not the conditions in Theorem* **??**.

*3) $Spin(2^k+1)$'s ($k \geq 4$) satisfy the conditions in Theorems* **??**, *but do not the conditions in Theorem* **??**.

*4) $E_6$, $E_7$, $E_8$ and $Spin(n)$ ($n \geq 10$ and $n \neq 2^*$) do not satisfy any of the conditions in Theorems* **??**, **??** *and* **??**.

We feel that it is too optimistic to think that $Spin(2^k+1)$'s ($k \geq 4$) satisfy the conditions in Theorem **??**, because they are not $A_\infty$-primitive but ($A_2$-)primitive (see Quillen [**?**]).

## 3. The proof of Theorem ??

It is clear that the condition iii) implies v). To show that iv) implies vii), we refer the following fact from [?]. .

**Fact 3.1.** *Let $\mu : G{\times}G \to G$ be the multiplication of the group $G$ and $T : G{\times}G \to G{\times}G$ be the switching mapping. Then we have the homotopy relation $j{\circ}\mu{\circ}T \sim j{\circ}\mu$, where $j$ denotes the inclusion $G \to B\Lambda G$.*

The condition v) implies that $j_*$ is injective. Hence we have vii) by the above fact.

## 4. The proof of Theorem ??

The condition c) implies clearly b) and the condition b) implies clearly a). Hence we are left to show that the condition a) implies c). By Fact ??, the condition c) implies that $G$ is homotopy commutative at the prime 2. By the Hubbuck's Torus Theorem [?], we have that $G$ must have the homotopy type of a torus at the prime 2.

## 5. The proof of Theorem ??

Firstly, it is well-known that viii) is equivalent with ix).

Secondly, we show that vi) is equivalent with vii): By the definition of $ad$, we have the equality

$$\mu = \mu(ad \times 1)(1 \times T)(\Delta \times 1)$$
$$\mu T = \mu(pr_2 \times 1)(1 \times T)(\Delta \times 1),$$

where $T$ denotes the transposition. Thus we have the following proposition:

**Proposition 5.1.** *1) For any $a \in H_*(G;\mathbb{F}_2)$ with $\Delta_* a = a{\otimes}1 + 1{\otimes}a + \sum_i a'_i{\otimes}a''_i$, the following equation holds.*

$$(\mu_* - \mu_* T_*)(a{\otimes}b) = (ad_* - pr_{2*})(a{\otimes}b) + \sum_i \mu_*(ad_*(a'_i{\otimes}b){\otimes}a''_i)$$

*2) If we make an additional assumption that every element $x$ in $H^*(G{\times}G;\mathbb{F}_2)$ satisfies $ad_*(x) = pr_{2*}(x)$ in dimensions $* < m$, then we have the following equation for any $w \in H_m(G{\times}G;\mathbb{F}_2)$.*

$$(\mu_* - \mu_* T_*)(w) = (ad_* - pr_{2*})(w).$$

Thus vi) is equivalent with vii).

Thirdly, we show that vii) is equivalent with viii): vii) is equivalent with the condition that $H^*(G;\mathbb{F}_2)$ is bicommutative and biassociative, which is equivalent with the condition that $H^*(G;\mathbb{F}_2)$ is primitively generated by Kane [?]. Thus we have that vii) is equivalent with viii). This completes the proof of Theorem **??**.

## 6. The proof of Theorem ??

Firstly, it is obvious that v) implies iv).

Thus we are left to show iv) implies v): Assuming iv), we consider the cohomology Eilenberg-Moore spectral sequence associated with the fibration (**??**). Then, by the standard argument of the Eilenberg-Moore spectral sequence, the $E_2$-term of the spectral sequence is described as

$$E_2^{*,*} = Cotor^{*,*}_{H^*(G;\mathbb{F}_2)}(\mathbb{F}_2, H^*(G;\mathbb{F}_2)).$$

converging to $H^*(B\Lambda G;\mathbb{F}_2)$. Hence iv) implies that $j_* : H_*(G;\mathbb{F}_2) \to H_*(B\Lambda G;\mathbb{F}_2)$ is injective, and hence, $E_2^{0,*} = \ker d_1 : H_*(B\Lambda G;\mathbb{F}_2) \to H_*(B\Lambda G;\mathbb{F}_2)\otimes H_*(B\Lambda G;\mathbb{F}_2)$ should be $H_*(B\Lambda G;\mathbb{F}_2)$, where $d_1$ is given by $ad_* - pr_{2*}$. Thus iv) implies v). This completes the proof of Theorem **??**.

## 7. The condition i) implies ii) and iii)

We shall show that the conditions i), ii) and iii) are equivalent.

In this section, we show that i) implies ii) and iii): i) implies that $H^*(G;\mathbb{Z})$ is an exterior algebra on odd primitive generators. Then the generators of $H^*(G;\mathbb{Z})$ are all transgressive and $H^*(BG;\mathbb{Z})$ is a polynomial algebra. Let $\ell : BG \to BK = BG_{\mathbb{Q}}$ be the rationalisation to the generalised Eilenberg-MacLane space where $K = \Omega BK \simeq G_{\mathbb{Q}}$:

$$\begin{array}{ccccccc}
G & = & G & \xrightarrow{\Omega(\ell)} & \Omega BK & = & \Omega BK \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\Lambda G & = & \Lambda BG & \xrightarrow{\Lambda(\ell)} & \Lambda BK & = & \{BK\}\times\Omega BK \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_{\mathbb{Q}}} & & \downarrow{\scriptstyle pr_1} \\
BG & = & BG & \xrightarrow{\ell} & BK & = & BK
\end{array} \tag{7.1}$$

Comparing the Serre spectral sequences for $p : \Lambda BG \to BG$ and for $pr_1 : \{BK\}\times K \to BK$, we can obtain that the spectral sequence collapses from the $E_2$-terms, and hence i) implies v). Again by i), every generators of $H^*(G;\mathbb{Z})$ are in odd dimensions, and hence the square of generators are all trivial in the integral cohomology. This implies that the module isomorphism gives an isomorphism of algebras. Thus the condition i) implies iii).

Also by (??), we have $\Lambda BG_{\mathbb{Q}} = BG_{\mathbb{Q}} \times \Omega BG_{\mathbb{Q}} \simeq BG_{\mathbb{Q}} \times G_{\mathbb{Q}}$ and hence the adjoint action of $G_{\mathbb{Q}}$ on $\Omega G_{\mathbb{Q}}$ is trivial up to homotopy. Since the integral homologies of $G$ and $\Omega G$ has no torsion, it is embedded in the rational homologies, and hence we get that the adjoint action of $G$ on $\Omega G$ induces trivial action on the homologies. Thus the condition i) implies ii).

## 8. Bockstein operation in primitively generated cohomology

Before proving the converse, we need to show the following

**Lemma 8.1.** *Let $H^*(G;\mathbb{F}_2)$ be primitively generated. If $G$ has 2-torsion, there is an odd primitive generator $y = j^*y'$ and a non-zero primitive element $x = j^*x'$ both of height 2, which satisfies $x' = Sq^1y'$.*

*Proof.* Since $H^*(G;\mathbb{F}_2)$ be primitively generated, $G$ has no higher torsion by Browder [?]. Let $x_1 = Sq^1y_1$ be the non-trivial Bockstein image appearing in the lowest dimension, say $N+1$, provided that $G$ has 2-torsion. Then by Browder [?], $N$ must be odd and $y_1$ is an indecomposable. Thus $y_1$ is an odd primitive generator of height $\geq 2$, and $x_1 = Sq^1y_1$ is a primitive element.

Using the Rothenberg-Steenrod spectral sequence for the group $G$, the following proposition is clear by the definition of $N$.

**Proposition 8.2.** *1) The module of primitive elements $P\tilde{H}^*(G;\mathbb{F}_2)$ is concentrated in odd dimensions up to dimension $N$.*

*2) $\tilde{H}^*(G;\mathbb{F}_2)$ and $\tilde{H}^*(G;\mathbb{F}_2)$ are exterior algebras on odd primitive generators up to dimension $N$.*

*3) $\tilde{H}^*(BG;\mathbb{F}_2)$ is concentrated in even dimensions up to dimension $N+1$.*

Now let us consider the Serre spectral sequence for the fibration $EG \times_G G \to BG$. In $E_2$-term, $1 \otimes y_1$ is primitive with respect to the fibrewise Hopf structure, since $y_1$ is primitive with respect to the Hopf structure of the group $G$. Then the differential image of $1 \otimes y_1$ has even total dimension and is in $\tilde{H}^*(BG;\mathbb{F}_2) \otimes P\tilde{H}^*(G;\mathbb{F}_2)$. It is impossible by 1) and 3) of Proposition ??. This implies that $1 \otimes y_1$ is a permanent cycle in the Serre spectral sequence. Hence the odd primitive generator $y_1$ is in the image of $j^*$ and has height $\geq 2$. If $y_1$ has height $> 2$, we get an odd primitive generator $y_2 = Sq^{N-1}y_1$ in the image of $j^*$ in a higher dimension which satisfies $Sq^1y_2 = Sq^Ny_1 = y_1^2 \neq 0$. If $y_2$ has also height $> 2$, we can continue this process and so on. But our group $G$ has the homotopy type of a finite complex, this process should end in a finitely many steps. Thus we get an odd primitive generator $y_k$ and $y_{k+1}$ in the image of $j^*$ with $0 \neq y_k^2 = Sq^1y_{k+1}$ and $y_{k+1}^2 = 0$. Letting $y = y_{k+1}$ and $x = Sq^1y$, we have also $x^2 = 0$. This completes the proof of Lemma **??**. □

## 9. The condition iii) implies i)

Secondly, we show that iii) implies i): iii) implies clearly v) and iv), and hence, vi), vii), viii) and ix). Assuming that $G$ has 2-torsion, we will be led to a contradiction. By Fact **??**, we have a homotopy $\phi : I \times G \times G \to B\Lambda G$ of $j(gh)$ and $j(hg)$. This enables us to define a Kudo-Araki operation $Q_1 : H_m(G;\mathbb{F}_2) \to H_{2m+1}(B\Lambda G;\mathbb{F}_2)$ so that an even primitive element in the image of $Sq^1j^*$ has 1-implication in $H^*(B\Lambda G;\mathbb{F}_2)$. By Lemma **??**, there is an odd primitive generator $y$ in the image of $j^*$ with $x = Sq^1y$ and $x$ and $y$ have height 2, if $G$ has 2-torsion. Thus $x'$ has 1-implication. Let $a$ be an even generator with $b = aSq^1$ is dual to $y$. Then by ix), $a^2 = 0$ and hence $x'^2$ is non-zero in $H^*(B\Lambda G;\mathbb{F}_2)$. Hence by $Sq^2y'^2 = x'^2$, $y'^2$ is non-zero for any choice of $y'$. This contradicts with iii), and hence, $G$ has no 2-torsion. Thus iii) implies i).

## 10. The condition ii) implies vi) through ix)

To show that ii) implies i), we need to show that ii) implies vi) through ix). Assuming that $H_*(G;\mathbb{F}_2)$ is not commutative, under the condition ii), we shall be led to a contradiction. Let $[a,b]$ be a non-zero commutator in the lowest dimension, say $m$. Since $H_*(G;\mathbb{F}_2)$ is associative, we have that $a$ and $b$ are generators and $[a,b]$ is primitive. Then by 2) of Proposition **??**, it follows that $ad_*(a\otimes b) = [a,b]$. Then by Theorem 5.4.1 (c) of [**?**], $m$ must be odd, and hence, we may assume that $a$ is even indecomposable and $b$ is odd indecomposable. Then there is an odd primitive element $y$ such that the Kronecker index $\langle b,y\rangle$ is non-zero. Since $H^*(G;\mathbb{F}_2)$ is associative and commutative, by Proposition 4.21 of [**?**], the odd primitive element $y \in PH^*(G;\mathbb{F}_2)$ is indecomposable. Hence we can choose an odd primitive generator $b'$ as a representative of the class $[b]$ in $QH_*(G;\mathbb{F}_2) = H_*(G;\mathbb{F}_2)/(\text{decomposable})$. Thus we have $ad_*(a\otimes b') = [a,b'] = [a,b] \neq 0$. By a series of results on the cohomology of a Hopf space such as [**?**], [**?**] or [**?**], it follows that $\sigma_* : QH_{even}(\Omega G;\mathbb{F}_2) \to PH_{odd}(G;\mathbb{F}_2)$ is surjective. Hence there is an element $b_0$ such that $\sigma_*(b_0) = b'$. By the relation $\sigma_*Ad_* = ad_*(1\times\sigma)_*$, we have

$$\sigma_*Ad_*(a\otimes b_0) = ad_*(1\otimes\sigma_*)(a\otimes b_0) = ad_*(a\otimes b') \neq 0.$$

This contradicts with $Ad_* = pr_{2*}$, and hence ii) implies vi).

Thus we have that ii) implies vi), vii) viii) and ix).

## 11. The existence of 1-implication

The presence of 2-torsion in $G$ implies 1-implication: Let us consider the subspaces of $B\Lambda G : G \overset{j_G}{\to} E^2G\times_G G \overset{i_2}{\to} (EG)\times_G G \simeq B\Lambda G$, where we denote by $E^2G$ the space $G * G/e * e$ obtained from $G * G$ by collapsing $e * e$. We remark here that, for any space $H$ which allow a left action of $G$ with a fixed point, there is a fibration $G \overset{j_H}{\to} E^2G\times_G H \overset{p_H}{\to} \Sigma G$ with cross-section $s_H : \Sigma G \to E^2G\times_G H$, i.e, a space over $\Sigma G$. Then the characteristic map $\psi_H : C(G)\times H \to E^2G\times_G H$ for the fibration $E^2G\times_G H \to \Sigma G$ is given as follows:

$$\psi_H(t, g; h) = (tg + (1 - t)e; h).$$

Thus $E^2G\times_G G$ is given by $G \cup_{ad} C(G)\times G$, and hence the cohomology of $E^2G\times_G G$ is isomorphic with $H^*(\Sigma G)\otimes H^*(G)$ as $H^*(\Sigma G)$-modules, since $ad^* = pr_2^*$ by vii).

The proof of Fact **??** still works well on the subspace $E^2G\times_G G$ and we have a homotopy $\phi_2 : I \times G \times G \to E^2G\times_G G$ so that $\phi = i_2\phi_2$:

$$\phi_2(t, g, h) = (tg + (1 - t)e; hg) = \psi_G(1\times 1\times\mu)(1\times T)(\Delta\times 1)(g, h). \tag{11.1}$$

By the arguments given in showing i) assuming iii), we have a homology operation $Q_1' : H_m(G; \mathbb{F}_2) \to H_{2m+1}(E^2G\times_G G; \mathbb{F}_2)$ so that we can get a relation $Q_1 = i_2Q_1'$ and that an even primitive element in the image of $Sq^1 j_G^*$ has 1-implication in $H^*(E^2G\times_G G; \mathbb{F}_2)$.

It follows from Proposition **??**, that there is an odd primitive generator $y$ of height 2 in the image of $j^* = j_G^* i_2^*$ with $x = Sq^1 y \neq 0$, if $G$ has 2-torsion. Then by the arguments given in showing i) assuming iii), we have that $i_2^* x'^2$ is non-trivial in $\bar{H}^*(E^2G\times_G G; \mathbb{F}_2)$, while $x^2 = 0$ in $H^*(G; \mathbb{F}_2)$. This implies that the non-trivial element $i_2^* x'^2 \in \bar{H}^*(E^2G\times_G G; \mathbb{F}_2)$ lies in both the kernel of $j_G^*$ and the image of $Sq^2 i_2^*$.

## 12. The representative cocycle of $i_2^* x'^2$

Let us consider the following exact sequence obtained from $E^2G\times_G H$ by collapsing $\Sigma G\vee H$:

$$\bar{H}^*(\Sigma G\wedge H; \mathbb{F}_2) \overset{q_H{}^*}{\to} \bar{H}^*(E^2G\times_G H; \mathbb{F}_2) \overset{\ell_H{}^*}{\to} \bar{H}^*(\Sigma G\vee H; \mathbb{F}_2), \tag{12.1}$$

where $\ell_H$ denotes the inclusion and $q_H$ the collapsion.

Let us recall that $j_G^* i_2^* x'^2 = x^2 = 0$. Also the restriction of $i_2^* x'^2$ to $\bar{H}^*(\Sigma G; \mathbb{F}_2)$ is zero, since a suspension space has no non-trivial cup-product in its cohomology. Then the exactness of (**??**) implies that $i_2^* x'^2$ is in the image of $q_G{}^*$. This implies the following proposition:

**Proposition 12.1.** *we may choose the representative cocyle $\eta = {q_G}^{\#}(\eta_0)$ of $i_2^* x'^2$, where $\eta_0$ is a cocycle in $C^{\#}(\Sigma G \wedge G)$.*

On the other hand, the Hopf structure $\mu : G\times G \to G$ induces the fibrewise Hopf structure on $E^2G\times_G G \to \Sigma G$, say $\hat{\mu} : (E^2G\times_G G)\times_{\Sigma G}(E^2G\times_G G) = E^2G\times_G(G\times G) \to E^2G\times_G G$. Let us denote by $\hat{pr}_t : E^2G\times_G(G\times G) \to E^2G\times_G G$ the projection to the $t$-th factor and by $\hat{in}_t : E^2G\times_G G \to E^2G\times_G(G\times G)$ the inclusion to the $t$-th factor of fibrewise Hopf spaces. Then the primitivity of $y$ implies that

$$\hat{\mu}^*(i_2^*y') = \hat{pr}_1{}^*(i_2^*y') + \hat{pr}_2{}^*(i_2^*y') + \sum_i a_i \otimes b_i \otimes c_i,$$

where $a_i$ is in $\tilde{H}^*(\Sigma G;\mathbb{F}_2)$. Since $G$ has no 2-torsion in its cohomology up to dimension N, $E^2G\times_G G$ and $E^2G\times_G G\times G$ have no 2-torsion, too. Thus we have

$$\hat{\mu}^*(i_2^*x') = \hat{pr}_1{}^*(i_2^*x') + \hat{pr}_2{}^*(i_2^*x'),$$

and since $\hat{\mu}$ induces a map $\hat{\mu}' : \Sigma G\wedge(G\times G) \to \Sigma G\wedge G$, we have

$$\begin{aligned} q_{G\times G}^*\hat{\mu'}^*([\eta_0]) &= \hat{\mu}^*(q_G^*[\eta_0]) = \hat{pr}_1{}^*(i_2^*x') + \hat{pr}_2{}^*(i_2^*x'), = \hat{pr}_1{}^*(q_G^*[\eta_0]) + \hat{pr}_2{}^*(q_G^*[\eta_0]), \\ &= q_{G\times G}^*(\hat{pr}_1{}^*([\eta_0]) + \hat{pr}_2{}^*([\eta_0])), \end{aligned}$$

By vi), we have that $q_{G\times G}^*$ is injective, and hence $[\eta_0]$ satisfies

$$\hat{\mu'}^*([\eta_0]) = \hat{pr}_1{}^*([\eta_0]) + \hat{pr}_2{}^*([\eta_0])$$

This yields the following proposition:

**Proposition 12.2.**

$$\hat{\mu'}^{\#}(\eta_0) = \hat{pr}_1{}^{\#}(\eta_0) + \hat{pr}_2{}^{\#}(\eta_0) + Im\ \delta.$$

## 13. Computations on Kudo-Araki operation and Browder's operation

Since the base space $\Sigma G$ has a cross-section, the condition vi) implies that $q_G^*$ is injective. Now we show the following lemma.

**Lemma 13.1.** *There is an element $[\eta_0] \in \bar{H}^*(\Sigma G\wedge G;\mathbb{F}_2)$ which satisfies that $i_2^*x'^2 = q_G^*[\eta_0]$ and that the Kronecker Index $\langle(\Sigma a)\otimes b, [\eta_0]\rangle$ is non-zero, where $aSq^1 = b$ is a primitive generator dual to $y$.*

*Proof.* Let us recall that $\phi_2(t,g,h) = \psi_G(t,g,\mu(h,g))$, and hence, we have

$$\begin{aligned}
\phi_{2\#}(e^1\otimes\alpha\otimes\beta) &= \psi_{G\#}(1\times 1\times\mu)_\#(1\times 1\times T)_\#(1\times\Delta\times 1)_\#(e^1\otimes\alpha\otimes\beta) \\
&= \psi_{G\#}(e^1\otimes\alpha\otimes\beta) \pm \sum_i \psi_{G\#}(e^1\otimes\alpha'_i\otimes\beta\cdot\alpha''_i), \\
&= \psi_{G\#}(e^1\otimes\alpha\otimes\beta) \pm \sum_i \psi_{G\#}(1\otimes 1\otimes\mu)_\#(e^1\otimes\alpha'_i\otimes\beta\cdot\alpha''_i), \\
&= \psi_{G\#}(e^1\otimes\alpha\otimes\beta) \pm \sum_i \hat{\mu}_\#\psi_{G\#}(e^1\otimes\alpha'_i\otimes\beta\otimes\alpha''_i),
\end{aligned}$$

where we denote $\Delta_\#(\alpha) = \alpha\otimes 1 + \sum_i \alpha'_i\otimes\alpha''_i$ with $a''_i$'s degree $> 0$.

Following to Browder, let $\alpha$ and $\beta$ be the representing chains in $C_\#(G;\mathbb{F}_2)$of $a$ and $b = aSq^1$ which are dual to $x = Sq^1 y$ and $y$, respectively. We can take $\alpha$ and $\beta$ satisfying $\partial(\alpha) = 2\beta$ and $\partial(\beta) = 0$. Then we have $\partial(\alpha^2) = 2\beta\alpha + 2\alpha\beta$ which is zero modulo 2. Since ii) implies ix), we have that $\alpha^2$ represents zero, and hence, one can take $\gamma$ such that $\alpha^2 = 2\gamma + \mathit{Im}\ \partial$. Then it follows that $\partial(\gamma) = \beta\alpha + \alpha\beta$ and $\partial(j_{G\#}\gamma + \phi_{2\#}(e^1\otimes\alpha\otimes\beta)) = 2j_{G\#}(\alpha\beta) - 2\phi_{2\#}(e^1\otimes\alpha\otimes\alpha)$ which is zero modulo 2. According to the arguments given in the proof of Browder's implication theorem, we get the following equation:

$$\begin{aligned}
1 &= \langle [j_{G\#}(\alpha\beta) - \phi_{2\#}(e^1\otimes\beta\otimes\beta)], i_2^*(x'y')\rangle \\
&= \langle [j_{G\#}\gamma + \phi_{2\#}(e^1\otimes\alpha\otimes\beta)], i_2^* x'^2\rangle \\
&= \langle j_{G\#}\gamma + \phi_{2\#}(e^1\otimes\alpha\otimes\beta), \eta\rangle \quad \mod 2
\end{aligned}$$

Since $\eta$ annihilates the image of $j_{G\#}$, we get

$$\langle \phi_{2\#}(e^1\otimes\alpha\otimes\beta), \eta\rangle = 1 \quad \mod 2.$$

By (**??**), we have $\phi_{2\#}(e^1\otimes\alpha\otimes\beta) = \psi_{G\#}(e^1\otimes\alpha\otimes\beta) \pm \sum_i \psi_{G\#}(e^1\otimes\alpha'_i\otimes\mu_\#(\beta\otimes\alpha''_i))$. Then by Proposition **??**, the following equation modulo 2 follows:

$$\begin{aligned}
1 &= \langle \phi_{2\#}(e^1\otimes\alpha\otimes\beta), \eta\rangle \\
&= \langle \psi_{G\#}(e^1\otimes\alpha\otimes\beta), {q_G}^\#\eta_0\rangle \pm \sum_i \langle \psi_{G\#}(e^1\otimes\alpha'_i\otimes\mu_\#(\beta\otimes\alpha''_i)), \eta\rangle
\end{aligned}$$

Since $\psi_G$ satisfies the equation $\psi_G(t,g,\mu(h_1,h_2)) = \hat{\mu}(\psi_G(t,g,h_1), \psi_G(t,g,h_2))$, we get

$$
\begin{aligned}
1 &= \langle q_{G\#}\psi_{G\#}(e^1\otimes\alpha\otimes\beta), \eta_0\rangle \pm \sum_i \langle \psi_{G\#}(1\times 1\times\mu)_\#(e^1\otimes\alpha_i'\otimes\beta\otimes\alpha_i''), q_G{}^\#\eta_0\rangle \\
&= \langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle \pm \sum_i \langle q_{G\#}\hat{\mu}_\#\psi_{G\times G\#}(e^1\otimes\alpha_i'\otimes\beta\otimes\alpha_i''), \eta_0\rangle \\
&= \langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle \pm \sum_i \langle \hat{\mu}'{}_\# q_{G\times G\#}\psi_{G\times G\#}(e^1\otimes\alpha_i'\otimes\beta\otimes\alpha_i''), \eta_0\rangle \\
&= \langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle \pm \sum_i \langle (\Sigma\alpha_i')\otimes\beta\otimes\alpha_i'', \hat{\mu}'{}^\#\eta_0\rangle
\end{aligned}
$$

By Proposition **??**, we can proceed as

$$
\begin{aligned}
&= \langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle \pm \sum_i \langle (\Sigma\alpha_i')\otimes\beta\otimes\alpha_i'', \hat{pr}_1{}^\#\eta_0 + \hat{pr}_2{}^\#\eta_0 + Im\ \delta\rangle \\
&= \langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle \pm \sum_i \langle (\Sigma\alpha_i')\otimes\beta\otimes\alpha_i'', Im\ \delta\rangle
\end{aligned}
$$

Here, let us recall that $\beta$, $a_i'$'s and $a_i''$'s are cycles modulo 2. Thus we get

$$
\langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle = 1. \tag{13.1}
$$

It then follows that the Kronecker index $\langle (\Sigma\alpha)\otimes\beta, \eta_0\rangle$ is non-trivial modulo 2. Let us recall that $a$ and $b$ are the homology class of $\alpha$ and $\beta$ and that $\partial(\alpha) = 2\beta$. Then we have that $aSq^1 = b$ and the Kronecker index $\langle (\Sigma a)\otimes b, [\eta_0]\rangle$ is non-trivial. Also by vi), we have $i_2^* x'^2 = q_G^*[\eta_0]$ and $q_G^*$ is injective. This completes the proof of Lemma **??** □

## 14. Pulling-Back to $i_2^* y'^2$

Since $y^2 = 0$, we have that $i_2^* y'^2$ is in the image of $q_G^*$, say $i_2^* y'^2 = q_G^*[\epsilon_0)$. The squaring relation $Sq^1 y' = x'$ implies that $Sq^2 y'^2 = x'^2 = q_G^*[\eta_0]$ and $Sq^2 i_2^* y'^2 = i_2^* x'^2$, and hence $i_2^* y'^2 \neq 0$. Since $q_G^*$ is injective, $Sq^2[\epsilon_0] = [\eta_0]$. Then by Lemma **??**, it follows that

$$
\begin{aligned}
1 &= \langle ((\Sigma a)\otimes b), [\eta_0]\rangle = \langle ((\Sigma a)\otimes b), Sq^2[\epsilon_0]\rangle = \langle ((\Sigma a)\otimes b)Sq^2, [\epsilon_0]\rangle \\
&= \langle ((\Sigma a)Sq^2\otimes b + (\Sigma a)Sq^1\otimes bSq^1 + (\Sigma a)\otimes bSq^2), [\epsilon_0]\rangle,
\end{aligned}
$$

Here we have $aSq^1 = b$ and $bSq^1 = 0$. The elements $b$ and $bSq^2$ give odd primitive elements in $\bar{H}_*(G;\mathbb{F}_2)$. Since every non-zero primitive element in $\bar{H}_*(G;\mathbb{F}_2)$ is in the image of $\sigma_* : \bar{H}_*(\Sigma\Omega G;\mathbb{F}_2) \to \bar{H}_*(G;\mathbb{F}_2)$, and hence there is an element $w \in \bar{H}_*(\Sigma G\wedge\Sigma\Omega G;\mathbb{F}_2)$ such that $((\Sigma a)\otimes b)Sq^2 = (1\wedge\sigma)_* w$. This implies

$$
\langle (1\wedge\sigma)_* w, [\epsilon_0]\rangle = \langle ((\Sigma a)\otimes b)Sq^2, [\epsilon_0]\rangle = \langle ((\Sigma a)\otimes b), Sq^2[\epsilon_0]\rangle = \langle ((\Sigma a)\otimes b), [\eta_0]\rangle \neq 0.
$$

Thus we have shown the following proposition:

**Proposition 14.1.** *1) The element $w \in \bar{H}_*(\Sigma G \wedge \Sigma\Omega G;\mathbb{F}_2)$ satisfies $((\Sigma a)\otimes b)Sq^2 = (1\wedge\sigma)_* w$.*

*2) The class $[\epsilon_0] \in \bar{H}_*(\Sigma G \wedge \Sigma\Omega G;\mathbb{F}_2)$ satisfies $\langle w, (1\wedge\sigma)^*[\epsilon_0]\rangle \neq 0$.*

## 15. The condition ii) implies i)

Now we proceed the final step: Let us consider the decomposition of space $E^2G\times_G\Sigma\Omega G$:

$$E^2G\times_G\Sigma\Omega G = \Sigma\Omega G \cup_{\hat{Ad}} CG\times\Sigma\Omega G = (\Sigma G\vee\Sigma\Omega G)\cup_{\Psi}(CG\times C\Omega G),$$

where $\hat{Ad}$ is given by $\hat{Ad}(g, t\wedge\ell) = t\wedge Ad(g,\ell)$ and $\Psi : G*\Omega G \to \Sigma G\vee\Sigma\Omega G$ is the attaching map.

A direct computation shows that $\Psi \simeq [\iota_{\Sigma G}, \iota_{\Sigma\Omega G}] + in_2 H(Ad)$, and hence, $pr_2\Psi \simeq H(Ad)$, where $[\iota_{\Sigma G}, \iota_{\Sigma\Omega G}]$ denotes the Whitehead product and $H(Ad)$ the Hopf construction of $Ad : G\times\Omega G \to \Omega G$.

So we have the following commutative ladder of exact sequences:

$$\begin{array}{ccccccc}
\bar{H}^*(\Sigma G\wedge G;\mathbb{F}_2) & \xrightarrow{q_{\Sigma\Omega G}{}^*} & \bar{H}^*_{\Sigma G}(E^2G\times_G G;\mathbb{F}_2) & \xrightarrow{j_G{}^*} & \bar{H}^*(G;\mathbb{F}_2) & & \\
\| & & \downarrow & & \downarrow & & \\
\bar{H}^*(\Sigma G\wedge G;\mathbb{F}_2) & \xrightarrow{q_G{}^*} & \bar{H}^*(E^2G\times_G G;\mathbb{F}_2) & \xrightarrow{\ell_G{}^*} & \bar{H}^*(\Sigma G\vee G;\mathbb{F}_2) & & \\
\downarrow{\scriptstyle (1\wedge\sigma)^*} & & \downarrow{\scriptstyle (1\times_G\sigma)^*} & & \downarrow{\scriptstyle (1\vee\sigma)^*} & & \\
\bar{H}^*(\Sigma G\wedge\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{q_{\Sigma\Omega G}{}^*} & \bar{H}^*(E^2G\times_G\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{\ell_{\Sigma\Omega G}{}^*} & \bar{H}^*(\Sigma G\vee\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{\Psi^*} & \bar{H}^*(G*\Omega G;\mathbb{F}_2) \\
\| & & \uparrow & & \uparrow & & \| \\
\bar{H}^*(\Sigma G\wedge\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{q_{\Sigma\Omega G}{}^*} & \bar{H}^*_{\Sigma G}(E^2G\times_G\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{j_{\Sigma\Omega G}{}^*} & \bar{H}^*(\Sigma\Omega G;\mathbb{F}_2) & \xrightarrow{H(Ad)^*} & \bar{H}^*(G*\Omega G;\mathbb{F}_2)
\end{array}$$

where $j_{\Sigma\Omega G}$ denotes the inclusion of fibre and $\bar{H}^*_{\Sigma G}(-;\mathbb{F}_2) = \bar{H}^*(-,\Sigma G;\mathbb{F}_2)$ is the cohomology of spaces over $\Sigma G$.

By the commutativity of the above diagram, we have

$$(1\times_G\sigma)^* i_2^* y'^2 = (1\times_G\sigma)^* q_G^*[\epsilon_0] = q_{\Sigma\Omega G}^*(1\wedge\sigma)^*[\epsilon_0]$$

and 2) of Proposition **??** says that$(1\wedge\sigma)^*[\epsilon_0]$ is non-zero. Since $H(Ad)^*$ is essentially given by $Ad^* - pr_2^*$, it follows that $q_{\Sigma\Omega G}^*$ is injective, if ii) holds. Thus by ii), we have

$$(1\times_G\sigma)^* i_2^* y'^2 \neq 0. \tag{15.1}$$

Take $y'' = i_2^* y - p_G^* s_G^* i_2^* y \in \bar{H}^*_{\Sigma G}(E^2G\times_G G;\mathbb{F}_2)$, where we denote by $p_H$ the projection $E^2G\times_G H \to \Sigma G$ and $s_H$ the cross-section of $p_H$. Then it follows that $H(Ad)^*(\sigma^* y) = H(Ad)^*\sigma^* j_G^* y'' = H(Ad)^* j_{\Sigma\Omega G}^*(1\times_G\sigma)^* y'' = 0$ and $(1\times_G\sigma)^* y'' \in \bar{H}^*_{\Sigma G}(E^2G\times_G\Sigma\Omega G;\mathbb{F}_2)$.

Then the functional cup product theorem of Thomas [?] tells us

$$(1\times_G\sigma)^*y''^2 = q^*_{\Sigma\Omega G}(y_1\otimes y_2),$$

when $Ad^*y = y_1\otimes 1 + 1\otimes y_2$. Here we have $y_1 = in_1^*Ad^*y = 0$ and $y_2 = in_2^*Ad^*y = y$, and hence

$$(1\times_G\sigma)^*y''^2 = q^*_{\Sigma\Omega G}(0\otimes y) = 0.$$

Then by $i_2^*y' = y'' + Im\ p^*$, it follows that

$$(1\times_G\sigma)^*i_2^*y'^2 = 0. \tag{15.2}$$

It's a contradiction. Hence $q^*_{\Sigma\Omega G}$ must have non-trivial kernel and $H(Ad)^*$ must be non-trivial. This implies that $Ad^* \neq pr_2^*$ and the image of $Ad^* - pr_2^*$ should contain non-zero elements $w\otimes\sigma^*u +$ other terms and $x\otimes\sigma^*y +$ other terms. This completes the proof of Theorem **??**.

## 16. Examples

Our observation covers some results obtained in [?] and [?]:

**Example 16.1.** *We obtain the following relations in the modulo 2 cohomologies.*

*(1) For $G = G_2$ the 1-connected compact connected exceptional Lie group of type $G_2$, our observation implies ${x'_3}^4 \neq 0$ and ${x'_5}^2 \neq 0$, and hence*

$$(Ad^* - pr_2^*)u_{10} = x_3^2\otimes u_4,$$
$$(Ad^* - pr_2^*)u_8 \ = x_3^2\otimes u_2.$$

*(2) For $G = E_6$ the 1-connected compact connected exceptional Lie group of type $E_6$, our observation implies*

$$(ad^* - pr_2^*)x_{15} = (\mu^* - T^*\mu^*)x_{15} = x_3^2\otimes x_9 + x_9\otimes x_3^2,$$

*and hence we obtain*

$$(Ad^* - pr_2^*)u_{14} = x_3^2\otimes u_8,$$

*and also by ${x'_3}^4 \neq 0$ and ${x'_5}^2 \neq 0$, we have*

$$(Ad^* - pr_2^*)u_{10} = x_3^2\otimes u_4,$$
$$(Ad^* - pr_2^*)u_8 \ = x_3^2\otimes u_2.$$

*(3) For $G = \Omega Z$ the Dwyer-Wilkerson complex with $H^*(\Omega Z;\mathbb{F}_2) = \mathbb{F}_2[x_7]/(x_7^4)\otimes\Lambda(x_{11}, x_{13})$, our observation implies ${x'_7}^4 \neq 0$ and ${x'_{13}}^2 \neq 0$, and hence*

$$(Ad^* - pr_2^*)u_{26} = x_7^2\otimes u_{12},$$

$$(Ad^* - pr_2^*)u_{24} = x_3^2\otimes u_{10}.$$

*Since $x_{13} = Sq^2 x_{11}$, we also have ${x'_{13}}^2 = Sq^4{x'_{11}}^2 \neq 0$, and hence*

$$(Ad^* - pr_2^*)u_{20} = x_7^2\otimes u_8.$$

## References


[1] S. Araki and T. Kudo, *Topology of $H_n$-spaces and H-squaring operations*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **10** (1956), 85–120.
[2] W. Browder, *Torsion in Hopf space*, Ann. Math. **74** (1961), 24–51.
[3] W. Browder, *Homotopy commutative H-space*, Ann. Math. **75** (1962), 283–311.
[4] W. Browder, *On differential Hopf algebras*, Trans. Amer. Math. Soc. **107** (1963), 153–176.
[5] W. Browder, *Higher torsion in H-spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.
[6] A. Clark, *Homotopy commutativity and the Moore spectral sequence*, Pacific. J. Math. **15** (1965), 65–74.
[7] A.L. Cook and M.C. Crabb, *Fibrewise Hopf structures on sphere-bundles*, J. London Math. Soc. **48** (1993), 365–384.
[8] J. Hubbuck, *On homotopy commutative H-spaces*, Topology **8** (1969), 119-126
[9] N. Iwase, *Adjoint action of a finite loop space*, Proc. Amer. Math. Soc. **125** (1997), 2753–2757.
[10] R. Kane, *Primitivity and finite H-spaces*, Q. J. Math. Oxford **3** (1975), 185–197.
[11] K. Kono and K. Kozima, *The adjoint action of a Lie group on the space of loops*, J. Math. Soc. Japan **45** (1993), 495–509.
[12] K. Kono and K. Kozima, *The adjoint action of the Dwyer-Wilkerson H-space on its loop space*, J. Math. Kyoto U. **35** (1995), 53–62.
[13] J.P. Lin, *Torsion in H-spaces II*, Ann. Math. **107** (1978), 41–88.
[14] J. Milnor and J.C. Moore, *On the structure of Hopf Algebras*, Ann. Math. **81** (1965), 211–264.
[15] J.C. Moore, *La Suspension*, Seminar H. Cartan **12** (1959/60), expose 6.
[16] D. Quillen, *The mod 2 cohomology rings of extra-special 2-groups and the spinor groups*, Math. Ann. **194** (1971), 197–212.
[17] E. Thomas, *On functional cup-products and the transgression operator*, Arch. Math. **12** (1961), 435–444.
[18] A. Zabrodsky, *Implications in the cohomology of H-space*, Illinois J. Math. **16** (1971), 363–375.