Canadian Mathematical Bulletin
doi:10.4153/CMB-2011-130-8

The Milnor–Stasheff Filtration on Spaces and Generalized Cyclic Maps

Norio Iwase, Mamoru Mimura, Nobuyuki Oda, and Yeon Soo Yoon

Abstract. The concept of C_k-spaces is introduced, situated at an intermediate stage between H-spaces and T-spaces. The C_k-space corresponds to the k-th Milnor–Stasheff filtration on spaces. It is proved that a space X is a C_k-space if and only if the Gottlieb set $G(Z,X) = [Z,X]$ for any space Z with $\operatorname{cat} Z \leq k$, which generalizes the fact that X is a T-space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space B. Some results on the C_k-space are generalized to the C_k^f-space for a map $f\colon A \to X$. Projective spaces, lens spaces and spaces with a few cells are studied as examples of C_k-spaces, and non-C_k-spaces.

1 Introduction

A 0-connected space X is called a T-*space* if the fibration $\Omega X \to X^{S^1} \to X$ is fiber homotopically trivial [1], and it is known that any 0-connected H-space is a T-space. To investigate intermediate stages between H-spaces and T-spaces, Aguadé [1] defined T_k-spaces for any integer $k \geq 1$ and $k = \infty$, making use of the Milnor–Stasheff filtration on spaces, so that the T_∞-space is an H-space and the T_1-space is a T-space. It seems that relations between T_k-spaces and the L-S category of spaces were not investigated clearly after his work. In this paper we define the concept of the C_k-space for $k \geq 1$ so that the C_1-space is the same as the T-space and the C_∞-space is an H-space. We also employ the Milnor–Stasheff filtration on spaces to define C_k-spaces. However, the definition of the C_k-space is directly connected with the L-S category; it enables us to prove, for example, that a space X is a C_k-space if and only if the Gottlieb set $G(Z,X) = [Z,X]$ for any space Z with $\operatorname{cat} Z \leq k$ (Theorem 2.3), which is a generalization of the fact that X is a T-space if and only if the Gottlieb group $G(\Sigma B, X) = [\Sigma B, X]$ for any space B [26, Theorem 2.2].

For each k, let $j_k^X\colon \Sigma\Omega X = P^1(\Omega X) \to P^k(\Omega X)$ and $e_k^X\colon P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusions for the spaces $P^k(\Omega X)$ [16, 21] (see §2). Let $f\colon A \to X$ be any map. A 0-connected space X is called a C_k^f-*space* if $e_k^X\colon P^k(\Omega X) \to X$ is f-cyclic (Definition 3.1). A $C_k^{1_X}$-space X is called a C_k-*space* (Definition 2.1).

We show that a space X is a C_k^f-space if and only if $G^f(Z,X) = [Z,X]$ for any space Z with $\operatorname{cat} Z \leq k$ (Theorem 3.2). Let $f\colon A \to X$ and $g\colon B \to Y$ be any maps. The product space $X \times Y$ is a $C_k^{f\times g}$-space if and only if X is a C_k^f-space and Y is a C_k^g-space (Theorem 4.7). It follows that the product space $X \times Y$ is a C_k-space if and only if both X and Y are C_k-spaces (Theorem 4.8).

Received by the editors August 6, 2009.
Published electronically June 29, 2011.
The first and third authors were partly supported by JSPS Grant-in-Aid for Scientific Research (No. 19540106).
AMS subject classification: **55P45**, 55P35.
Keywords: Gottlieb sets for maps, L-S category, T-spaces.

Let $\widetilde{X}$ be a covering space of a space X with the covering map $p\colon \widetilde{X} \to X$ and $1 \leq k \leq \infty$. Let $f\colon A \to X$, $\widetilde{f}\colon B \to \widetilde{X}$, and $q\colon B \to A$ be maps such that the following diagram is homotopy commutative,

$$\begin{array}{ccc} B & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & X \end{array}$$

In Theorem 4.9 we show that if X is a C_k^f-space, then the covering space $\widetilde{X}$ is a $C_k^{\widetilde{f}}$-space. A relation between two "multiplications" that are induced by a pairing and a copairing [18, Proposition 3.4] will be used to prove Theorem 4.9. A similar result holds for the T_k^f-space, which is a generalization of Aguadé's T_k-space (see Definition 3.3). If we put $f = 1_X$, $\widetilde{f} = 1_{\widetilde{X}}$, $q = p$, then we see that any covering space of a C_k-space (resp. Aguadé's T_k-space) is a C_k-space (resp. T_k-space) for any $1 \leq k \leq \infty$ (Theorem 4.10).

In the last section we study projective spaces, lens spaces and spaces with a few cells.

2 C_k-Spaces

We work in the category of topological spaces with base point. The symbol $f \sim g\colon X \to Y$ means the based homotopy relation and the symbol $X \simeq Y$ the based homotopy equivalence. The set of based homotopy classes of maps $[f]\colon X \to Y$ is denoted by $[X,Y]$. Let $f\colon A \to X$ be a map. A based map $g\colon B \to X$ is said to be *f-cyclic* [17] if there exists a map $\phi\colon B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow{f \vee g} & X \vee X \end{array}$$

is homotopy commutative, where $j\colon A \vee B \to A \times B$ is the inclusion and $\nabla\colon X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of an f-cyclic map g.

Clearly, g is f-cyclic if and only if f is g-cyclic. We write $f \perp g$ if g is f-cyclic. If $f \perp g$ for maps $f\colon A \to X$ and $g\colon B \to X$, then $(w \circ f \circ f') \perp (w \circ g \circ g')$ for any maps $w\colon X \to W$, $f'\colon A' \to A$, and $g'\colon B' \to B$ by [17, Theorems 1.4 and 1.5]. This formula is used repeatedly in the following arguments without further reference. A based map $g\colon B \to X$ is said to be *cyclic* [23] if $1_X \perp g$, that is, g is 1_X-cyclic. The *Gottlieb set* denoted by $G(B,X)$ is the set of all homotopy classes of cyclic maps from B to X.

The loop space ΩX of any space X has a homotopy type of an associative H-space. A 0-connected space X is filtered by the projective spaces of ΩX [16, 21]:

$$* = P^0(\Omega X) \hookrightarrow \Sigma\Omega X = P^1(\Omega X) \hookrightarrow \cdots \hookrightarrow P^k(\Omega X) \hookrightarrow \cdots \hookrightarrow P^\infty(\Omega X) \simeq X.$$

For each k, let $j_k^X\colon \Sigma\Omega X = P^1(\Omega X) \to P^k(\Omega X)$ and $e_k^X\colon P^k(\Omega X) \to P^\infty(\Omega X) \simeq X$ be the natural inclusions. We write $e^X = e_1^X\colon \Sigma\Omega X = P^1(\Omega X) \to X$. We see that $j_\infty^X \sim e^X\colon \Sigma\Omega X \to X$ and $e_\infty^X \sim 1_X\colon X \to X$.

A 0-connected space X is called a T_k-*space* [1] if $1_X \perp \bar{e}_k$ for some extension $\bar{e}_k\colon P^k(\Omega X) \to X$ of $e^X\colon \Sigma\Omega X \to X$, that is, there exists a map $\phi_k\colon X \times P^k(\Omega X) \to X$ such that $\phi_k \circ j \circ (1_X \vee j_k^X) \sim \nabla \circ (1_X \vee e^X)\colon X \vee \Sigma\Omega X \to X$. Aguadé showed that X is a T-space if and only if X is a T_1-space [1, Proposition 4.1]. If X is a T_k-space, then it is a T_i-space for any $1 \le i \le k$. By [1, Proposition 4.1(i)(ii)], a 0-connected space is an H-space if and only if it is a T_∞-space; we remark that $\bar{e}_\infty \sim 1_X$ when X is a 0-connected CW complex. The concepts of the T-space and the Gottlieb set are closely connected by the fact that X is a T-space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space B [26, Theorem 2.2].

Definition 2.1 Let $k \ge 1$ be an integer or $k = \infty$. A 0-connected space X is called a C_k-*space* if $1_X \perp e_k^X$, that is, the inclusion $e_k^X\colon P^k(\Omega X) \to X$ is cyclic. A 0-connected space X is called an NC-*space* if X is not a C_k-space for any $k \ge 1$.

Clearly any C_k-space is a T_k-space for any $k \ge 1$. We use the L-S category $\operatorname{cat} X$ for a 0-connected space X in the sense that $\operatorname{cat} X = n$ if n is the minimum number of categorical open coverings $U_0, U_1, \ldots, U_n$ of X, so that $\operatorname{cat} X = 0$ if and only if X is contractible and $\operatorname{cat} X \le 1$ if X is a suspension. Throughout this paper, we follow Iwase for the notations for the L-S category; his list of references covers much of the widely-known literature [11] .

We now recall Ganea's theorem [10, 11].

Theorem 2.2 (Ganea [3, 10]) *Let $k \ge 1$ be an integer or $k = \infty$ and assume that X is a 0-connected space. The category $\operatorname{cat} X \le k$ if and only if $e_k^X\colon P^k(\Omega X) \to X$ has a right homotopy inverse.*

In the rest of this section, we mention some results on the C_k-space that are obtained as special cases of the results on the C_k^f-spaces for a map $f\colon A \to X$ in the following sections, since the C_k-space is the C_k^f-space for the identity map $f = 1_X\colon X \to X$.

The property of the T-spaces in [26, Theorem 2.2] is extended to the C_k-spaces using the L-S category in the sense that the L-S category of any suspension space ΣB satisfies $\operatorname{cat} \Sigma B \le 1$.

Theorem 2.3 *Let $k \ge 1$ be an integer. A space X is a C_k-space if and only if $G(Z, X) = [Z, X]$ for any space Z with $\operatorname{cat} Z \le k$.*

Theorem 2.3 is a special case of Theorem 3.2 which is proved in the next section. The following proposition is a direct consequence of the definition.

Proposition 2.4 (i) *A space X is a T-space if and only if X is a C_1-space.*
(ii) *Any C_m-space is a C_n-space for $\infty \geq m \geq n \geq 1$.*
(iii) *A space X is an H-space if and only if X is a C_∞-space.*

As a direct consequence of Proposition 3.4(ii),(v) and Theorem 4.3, the following theorem is obtained.

Theorem 2.5 *Assume that* $\operatorname{cat} X = k \geq 1$. *Then X is an H-space if and only if X is a C_n-space for some $n \geq k$.*

It is known [14] that $\operatorname{cat} X \leq \dim X$ for any finite CW complex X. Thus, we obtain the following corollary.

Corollary 2.6 *If a T-space X is a 1-dimensional finite CW complex, then $X = S^1$.*

Example 2.7 By [1, Proposition 4.2] Aguadé obtained a space X such that X is a T_{p-1}-space but not a T_p-space. This space X is not a C_p-space, but it is not known whether X is a C_{p-1}-space or not.

3 C_k^f-Spaces for a Map $f\colon A \to X$

We denote the set of all homotopy classes of f-cyclic maps from B to X by

$$G(B; A, f, X) = G^f(B, X) = f^{\perp}(B, X) \subset [B, X].$$

This is called the *Gottlieb set for a map* $f\colon A \to X$. If $f = 1_X\colon X \to X$, then we recover the set $G(B, X)$ defined by Varadarajan [23]:

$$G(B, X) = G(B; X, 1_X, X) = G^{1_X}(B, X) = (1_X)^{\perp}(B, X).$$

In general, $G(B, X) \subset G^f(B, X) \subset [B, X]$ for any spaces A, B, X and any map $f\colon A \to X$. An example is shown in [27] such that $G(B, X) \neq G(B; A, f, X) \neq [B, X]$:

$$G_5(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_5(S^5, i_1, S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi_5(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Definition 3.1 Let $k \geq 1$ be an integer or $k = \infty$. Let $f\colon A \to X$ be any map. A 0-connected space X is called a C_k^f*-space* if $f \perp e_k^X$ (or $e_k^X\colon P^k(\Omega X) \to X$ is f-cyclic). A 0-connected space X is called an NC^f-space if X is not a C_k^f-space for any $k \geq 1$.

We see that a $C_k^{1_X}$-space X is a C_k-space.

Theorem 3.2 *Let $f\colon A \to X$ be any map. A space X is a C_k^f-space if and only if $G^f(Z, X) = [Z, X]$ for any space Z with* $\operatorname{cat} Z \leq k$.

Proof Suppose that X is a C_k^f-space, namely, $f \perp e_k^X$. Let Z be a space with $\operatorname{cat}\ Z \leq k$ and $g\colon Z \to X$ any map. Since $\operatorname{cat} Z \leq k$, there exists a map $s_k^Z\colon Z \to P^k(\Omega Z)$ such that $e_k^Z \circ s_k^Z \sim 1_Z$. We see that $e_k^X \circ P^k(\Omega g) \sim g \circ e_k^Z$ by the naturality of the construction of $P^k(\Omega Z)$, as is shown in the following homotopy commutative diagram:

$$\begin{array}{ccc} P^k(\Omega Z) & \xrightarrow{P^k(\Omega g)} & P^k(\Omega X) \\ {\scriptstyle e_k^Z}\downarrow & & \downarrow{\scriptstyle e_k^X} \\ Z & \xrightarrow{g} & X \end{array}$$

Hence the relation $f \perp e_k^X$ implies $f \perp (e_k^X \circ P^k(\Omega g) \circ s_k^Z)$ or $f \perp g$. It follows that $G^f(Z, X) = [Z, X]$.

Conversely, assume that $G^f(Z, X) = [Z, X]$ for any space Z with $\operatorname{cat} Z \leq k$. It is known that $\operatorname{cat} C_\theta \leq \operatorname{cat} Y + 1$ for any map $\theta\colon X \to Y$ [24, (1.6) Theorem, p. 459], where C_θ is the mapping cone of θ. Thus $\operatorname{cat} P^k(\Omega X) = \operatorname{cat} C_\theta \leq \operatorname{cat} P^{k-1}(\Omega X) + 1$, where $\theta\colon (\Omega X) * \cdots * (\Omega X)(k\text{-times}) \to P^{k-1}(\Omega X)$ is the map in [21, Part I, Theorem 12]. By induction, we have $\operatorname{cat} P^k(\Omega X) \leq k$. Thus we know that $e_k^X\colon P^k(\Omega X) \to X$ is f-cyclic by our assumption, and hence X is a C_k^f-space. ■

A space X is called an H^f-space for a map $f\colon A \to X$ if 1_X is f-cyclic (namely $f \perp 1_X$), and a T^f-space for a map $f\colon A \to X$ if $e^X\colon \Sigma\Omega X \to X$ is f-cyclic (namely $f \perp e^X$)[28, 29]. Any H-space X is an H^f-space and any H^f-space X is a T^f-space for any map $f\colon A \to X$. We remark that the 2-dimensional sphere S^2 is not an H-space nor a T-space, but it is an H^{η_2}-space and a T^{η_2}-space for the Hopf map $\eta_2\colon S^3 \to S^2$ [29, Example 2.10], [26, Corollary 2.8].

Definition 3.3 Let $f\colon A \to X$ be any map. A space X is called a T_k^f*-space* if $f \perp \bar{e}_k$ for some extension $\bar{e}_k\colon P^k(\Omega X) \to X$ of $e^X\colon \Sigma\Omega X \to X$, that is, there exists a map $\phi_k\colon A \times P^k(\Omega X) \to X$ such that $\phi_k \circ j \circ (1_X \vee j_k^X) \sim \nabla \circ (f \vee e^X)\colon A \vee P^1(\Omega X) \to X$.

An H^{1_X}-space X is an H-space and a $T_k^{1_X}$-space X is a T_k-space.

Proposition 3.4 *Let $f\colon A \to X$ be any map.*

(i) *X is a C_1^f-space $\Leftrightarrow$ X is a T_1^f-space $\Leftrightarrow$ X is a T^f-space.*

(ii) *Any C_m^f-space is a C_n^f-space for $\infty \geq m \geq n \geq 1$.*

(iii) *Any T_m^f-space is a T_n^f-space for $\infty \geq m \geq n \geq 1$.*

(iv) *If X is a C_k^f-space, then X is a T_k^f-space for $\infty \geq k \geq 1$.*

(v) *If X has the homotopy type of a CW complex, then the following equivalences hold:*

$$X \text{ is an } H^f\text{-space} \Leftrightarrow X \text{ is a } C_\infty^f\text{-space} \Leftrightarrow X \text{ is a } T_\infty^f\text{-space.}$$

Proof These results are direct consequences of the definitions except the following part of (v): "X is a T_∞^f-space $\Rightarrow$ X is an H^f-space", which is proved by a method similar to the proof of [1, Proposition 4.1 (ii)] as follows.

Suppose that X is a T_∞^f-space. Then $f \perp \bar{e}$ for some extension $\bar{e}\colon P^\infty(\Omega X)(\simeq X) \to X$ of $e_1^X\colon \Sigma\Omega X \to X$, and there exists a map $m\colon A \times P^\infty(\Omega X) \to X$ with axes f and $\bar{e}$, making the following diagram commutative up to homotopy:

$$
\begin{array}{ccccc}
A \times X & \xleftarrow[\simeq]{1\times e^X_\infty} & A \times P^\infty(\Omega X) & \xrightarrow{m} & X \\
& {}_{1\times e^X_1}\nwarrow & \cup & \nearrow & \\
& & A \times \Sigma\Omega X & &
\end{array}
$$

Let $g\colon X \to X$ be a map given by $g(x) = m \circ (1 \times e^X_\infty)^{-1}(*, x)$ for any $x \in X$. Then $g \sim \bar{e} \circ (e^X_\infty)^{-1}$ and we have $g \circ e^X_1 \sim e^X_1$, and hence $\Omega g \sim 1_{\Omega X}$ by taking adjoints. Then it follows that $g\colon X \to X$ is a weak homotopy equivalence and hence is a homotopy equivalence if X has the homotopy type of a CW complex, by a theorem of J. H. C. Whitehead, and there exists a map $h\colon X \to X$ such that $g \circ h \sim 1_X$. Hence we have $f \perp g$, which implies that $f \perp (g \circ h)$ or $f \perp 1_X$ by the composition formula we discussed at the start of Section 2. ∎

4 More about T^f_k-Spaces and C^f_k-Spaces

Proposition 4.1 *Let $f\colon A \to X$ and $g\colon B \to A$ be any maps.*

(i) *If X is an H^f-space, then X is an $H^{f\circ g}$-space.*
(ii) *If X is a T^f_k-space, then X is a $T^{f\circ g}_k$-space.*
(iii) *If X is a C^f_k-space, then X is a $C^{f\circ g}_k$-space.*

Proof The relations (i) $f \perp 1_X$, (ii) $f \perp \bar{e}_k$, and (iii) $f \perp e^X_k$ imply (i) $(f \circ g) \perp 1_X$, (ii) $(f \circ g) \perp \bar{e}_k$, and (iii) $(f \circ g) \perp e^X_k$, respectively, and we have the results. ∎

Proposition 4.2 *Assume that $f\colon A \to X$ has a right inverse $s\colon X \to A$,* i.e., *$f \circ s \sim 1_X$. Then the following results hold.*

(i) *An H^f-space X is an H-space.*
(ii) *A T^f_k-space X is a T_k-space.*
(iii) *A C^f_k-space X is a C_k-space.*

Proof These are immediate by Proposition 4.1. ∎

If X is an H^f-space, then X is a C^f_k-space for any $k \geq 1$ by Proposition 3.4 (ii), (v). The following theorem shows that the converse holds if $\operatorname{cat} X \leq k$.

Theorem 4.3 *Let $f\colon A \to X$ be any map.*

(i) *If X is a C^f_k-space and $\operatorname{cat} X \leq k$, then X is an H^f-space.*
(ii) *If X is a C_k-space and $\operatorname{cat} X \leq k$, then X is an H-space.*

Proof (i) Since $\operatorname{cat} X \leq k$, we see that $G^f(X, X) = [X, X]$ by Theorem 3.2. It follows that $f \perp 1_X$. (ii) is the case where $f = 1_X$, and hence $1_X \perp 1_X$. ∎

Theorem 4.4 *Assume that Y is a homotopy retract of X with the maps $r\colon X \to Y$ and $s\colon Y \to X$ such that $r \circ s \sim 1_Y$.*

(i) *If X is a C_k^f-space, then Y is a $C_k^{r\circ f}$-space for any map $f\colon A \to X$.*
(ii) *If X is a C_k-space, then Y is a C_k-space.*

Proof Let $\bar{r}_k = P^k(\Omega r)\colon P^k(\Omega X) \to P^k(\Omega Y)$ and $\bar{s}_k = P^k(\Omega s)\colon P^k(\Omega Y) \to P^k(\Omega X)$ be the maps induced by r and s, respectively. Then we see that

$$e_k^Y = r \circ s \circ e_k^Y = e_k^Y \circ \bar{r}_k \circ \bar{s}_k = r \circ e_k^X \circ \bar{s}_k\colon P^k(\Omega Y) \to Y.$$

Then (i) the relation $f \perp e_k^X$ implies $(r \circ f) \perp (r \circ e_k^X \circ \bar{s}_k)$, or $(r \circ f) \perp e_k^Y$ and (ii) the relation $1_X \perp e_k^X$ implies $(r \circ 1_X \circ s) \perp (r \circ e_k^X \circ \bar{s}_k)$, or $1_Y \perp e_k^Y$ [17, Theorems 1.4, 1.5]. ■

The following result is a generalization of Woo and Kim [25, Theorem 3.6].

Proposition 4.5 *Let $f\colon A \to X$ and $g\colon B \to Y$ be any maps. The relation*

$$G^{f\times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y)$$

holds for any space Z (under the identification $[Z, X \times Y] \cong [Z, X] \times [Z, Y]$).

Proof Let $\alpha\colon Z \to X$ and $\beta\colon Z \to Y$ be maps. We define a map $(\alpha, \beta)\colon Z \to X \times Y$ by $(\alpha, \beta) = (\alpha \times \beta) \circ \Delta_Z$ for the diagonal map $\Delta_Z\colon Z \to Z \times Z$. Suppose that $(\alpha, \beta) \in G^f(Z, X) \times G^g(Z, Y)$, which is identified with a map $(\alpha, \beta)\colon Z \to X \times Y$. Since $f \perp \alpha$ and $g \perp \beta$, we have $(f \times g) \perp (\alpha \times \beta)$ [17, Proposition 1.7]). It follows that $(f \times g) \perp \{(\alpha \times \beta) \circ \Delta_Z\}$ or $(f \times g) \perp (\alpha, \beta)$, and hence $(\alpha, \beta) \in G^{f\times g}(Z, X \times Y)$.

Conversely, suppose that $(\alpha, \beta) \in G^{f\times g}(Z, X \times Y)$ or $(f \times g) \perp (\alpha, \beta)$. Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ be the projections and $i_1\colon X \to X \times Y$ and $i_2\colon Y \to X \times Y$ be the inclusions defined by $i_1(x) = (x, y_0)$ and $i_2(y) = (x_0, y)$ for any $x \in X$ and $y \in Y$, where $x_0 \in X$ and $y_0 \in Y$ are base points. It follows that

$$\{p_1 \circ (f \times g) \circ i_1\} \perp \{p_1 \circ (\alpha, \beta)\} \quad \text{and} \quad \{p_2 \circ (f \times g) \circ i_2\} \perp \{p_2 \circ (\alpha, \beta)\}$$

and we have $f \perp \alpha$ and $g \perp \beta$. It follows that $\alpha \in G^f(Z, X)$ and $\beta \in G^g(Z, Y)$. ■

Remark 4.6 The converse of Proposition 1.7 of [17] holds by an argument similar to the proof of Proposition 4.5. Let $f_1\colon X_1 \to Z_1$, $f_2\colon X_2 \to Z_2$, $g_1\colon Y_1 \to Z_1$, $g_2\colon Y_2 \to Z_2$ be any maps. Then the following statements are equivalent.

(i) $f_1 \perp g_1$ and $f_2 \perp g_2$.
(ii) $(f_1 \times f_2) \perp (g_1 \times g_2)$

Theorem 4.7 *Let $f\colon A \to X$ and $g\colon B \to Y$ be any maps. The product space $X \times Y$ is a $C_k^{f\times g}$-space if and only if X is a C_k^f-space and Y is a C_k^g-space.*

Proof If $X \times Y$ is a $C_k^{f\times g}$-space, then for any space Z with $\operatorname{cat} Z \leq k$ we see

$$G^f(Z, X) \times G^g(Z, Y) \cong G^{f\times g}(Z, X \times Y) = [Z, X \times Y] = [Z, X] \times [Z, Y]$$

by Theorem 3.2 and Proposition 4.5, and hence $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$.

Conversely, suppose that X is a C_k^f-space and Y is a C_k^g-space. Then $G^f(Z, X) = [Z, X]$ and $G^g(Z, Y) = [Z, Y]$ for any space Z with $\operatorname{cat} Z \leq k$ by Theorem 3.2. It follows that $G^{f\times g}(Z, X \times Y) \cong G^f(Z, X) \times G^g(Z, Y) = [Z, X] \times [Z, Y] = [Z, X \times Y]$ for any space Z with $\operatorname{cat} Z \leq k$. ■

Theorem 4.8 *The product space $X \times Y$ is a C_k-space if and only if both X and Y are C_k-spaces.*

Proof Set $f = 1_X$ and $g = 1_Y$ in Theorem 4.7. Then we have the result. ∎

We now consider covering spaces of C_k^f-spaces and T_k^f-spaces.

Theorem 4.9 *Let $\widetilde{X}$ be a covering space of a space X with the covering map $p\colon \widetilde{X} \to X$ and $1 \leq k \leq \infty$. Let $f\colon A \to X$, $\widetilde{f}\colon B \to \widetilde{X}$, and $q\colon B \to A$ be maps such that the following diagram is homotopy commutative:*

$$\begin{array}{ccc} B & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & X \end{array}$$

(i) *If X is a C_k^f-space, then the covering space $\widetilde{X}$ is a $C_k^{\widetilde{f}}$-space.*

(ii) *If X is a T_k^f-space, then the covering space $\widetilde{X}$ is a $T_k^{\widetilde{f}}$-space.*

Proof (i) Since X is a C_k^f-space, there exists a map m_k for $f \perp e_k^X$. Consider the following diagram.

$$\begin{array}{ccc} B \times P^k(\Omega\widetilde{X}) & \xrightarrow{\widetilde{m}_k} & \widetilde{X} \\ {\scriptstyle q \times P^k(\Omega p)}\downarrow & & \downarrow{\scriptstyle p} \\ A \times P^k(\Omega X) & \xrightarrow{m_k} & X \end{array}$$

We must show that

$$(m_k \circ (q \times P^k(\Omega p))_*(\pi_1(B \times P^k(\Omega\widetilde{X})) \subset p_*\pi_1(\widetilde{X})$$

to obtain a map $\widetilde{m}_k\colon B \times P^k(\Omega\widetilde{X}) \to \widetilde{X}$ for $\widetilde{f} \perp e_k^{\widetilde{X}}$. Let $(\alpha, \beta) \in \pi_1(B \times P^k(\Omega\widetilde{X}))$ be any element. We see that

$$\begin{aligned} (m_k \circ (q \times P^k(\Omega p))_*((\alpha, \beta)) &= (f \circ q)_*(\alpha) \dot{+} (e_k^X \circ P^k(\Omega p))_*(\beta) \\ &= (p \circ \widetilde{f})_*(\alpha) + (p \circ e_k^{\widetilde{X}})_*(\beta) \\ &= p_*(\widetilde{f}_*(\alpha) + (e_k^{\widetilde{X}})_*(\beta)) \in p_*\pi_1(\widetilde{X}), \end{aligned}$$

by [18, Proposition 3.4 (1)], since $f \circ q \sim p \circ \widetilde{f}$ by assumption and the following diagram is homotopy commutative:

$$\begin{array}{ccc} P^k(\Omega\widetilde{X}) & \xrightarrow{e_k^{\widetilde{X}}} & \widetilde{X} \\ {\scriptstyle P^k(\Omega p)}\downarrow & & \downarrow{\scriptstyle p} \\ P^k(\Omega X) & \xrightarrow{e_k^{X}} & X \end{array}$$

(ii) is proved by an argument similar to (i); the proof is omitted. ■

The following theorem is obtained by setting $A = X$, $B = \widetilde{X}$, $q = p\colon \widetilde{X} \to X$, $f = 1_X$, and $\widetilde{f} = 1_{\widetilde{X}}$ in Theorem 4.9.

Theorem 4.10 *Any covering space of a C_k-space (resp. T_k-space) is a C_k-space (resp. T_k-space) for any $1 \leq k \leq \infty$.*

5 Applications and Examples

We have the following result by Theorem 2.5.

Proposition 5.1 *If X is a C_m-space with $\operatorname{cat} X \leq m$ for some $m \geq 1$, then X is an H-space.*

Proposition 5.2 (i) *If $\operatorname{cat} X = 1$ (for example, $X = \Sigma A$, or a general co-H-space) and X is not an H-space, then X is an NC-space.*
(ii) *If ΣX is a C_1-space, then $\Sigma X = S^1$, S^3, or S^7.*

Proof (i) and (ii) are obtained by Proposition 5.1. ■

Let X be a 0-connected space. A space X is called a *Gottlieb space* or a G-space if the Gottlieb group $G_m(X) = \pi_m(X)$ for any $m \geq 1$ [4, 5]. A space X is called a *Whitehead space* or a W-space if every Whitehead product $[\alpha, \beta] = 0$ in $[S^{m+n+1}, X]$ for any $\alpha \in [S^{n+1}, X]$, $\beta \in [S^{m+1}, X]$, and any $n, m \geq 0$. A space X is called a *generalized Whitehead space* or a GW-space if every generalized Whitehead product on X is trivial, that is, $[\alpha, \beta] = 0$ in $[\Sigma(A \wedge B), X]$ for any $\alpha \in [\Sigma A, X]$, $\beta \in [\Sigma B, X]$, and any spaces A, B.

Remark 5.3 The following implications hold:

(i) X is a C_1-space $\Rightarrow$ X is a G-space $\Rightarrow X$ is a W-space.
(ii) X is a C_1-space $\Rightarrow X$ is a GW-space $\Rightarrow X$ is a W-space.

(See [26, Theorem 2.2] and [20, Theorem 1.9] for (i); [12, Remark (4), p. 616] for (ii).)

The complex projective space CP^3 is a GW-space [12, Theorem 1] such that $\operatorname{cat}(CP^3) = 3$, but it is not a C_k-space for any k (Example 5.7). We note that CP^3 is not a G-space [20, Remark 3.4].

If $p > 2$, then $L^3(p)$ is a G-space, but it is not a C_k-space for any $k \geq 2$ (see Example 5.10 and Theorem 5.13).

Proposition 5.4 *Assume that X is a 1-connected space.*

(i) *X is a G-space $\Longrightarrow$ X is a rational H-space.*
(ii) *If $k \geq 1$, then the rationalization of any T_k-space (and hence any C_k-space) is an H-space.*

Proof (i) is obtained by Haslam [7] (see also [13, Theorem 3.4]). (ii) is a direct consequence of (i). ■

Example 5.5 It is known that H-spaces, T-spaces, and GW-spaces are equivalent in the class of spaces of L-S category ≤ 1 (see Propositions 2.4 , 5.1 and the definition of the GW-space). Then the following results hold by Proposition 3.4(v) and Theorem 4.3(ii).

(i) S^1, S^3 ,and S^7 are H-spaces and hence C_k-spaces for any $k \geq 1$.
(ii) If $1 \leq n < \infty$ and $n \neq 1, 3, 7$, then S^n is not an H-space and hence an NC-space, since $\operatorname{cat} S^n = 1$.

In the following argument we consider projective spaces RP^n, CP^n, and lens spaces $L^n(p)$ ($p \geq 2$); however, the cases RP^∞, CP^∞, and $L^\infty(p)$ are not referred to, since they are H-spaces and hence C_k-spaces for any $1 \leq k \leq \infty$.

Example 5.6 If $1 \leq n < \infty$ and $n \neq 1, 3, 7$, then the real projective space RP^n is an NC-space by Example 5.5(ii)and Theorem 4.10. However, RP^1, RP^3, and RP^7 are H-spaces and hence C_k-spaces for any $1 \leq k \leq \infty$.

Example 5.7 If a 1-connected space X is not a rational H-space, then X is an NC-space by Proposition 5.4. For $1 \leq n < \infty$, the complex projective space CP^n is not a rational H-space, and hence it is an NC-space.

Let S^{2n+1} be the unit sphere in the $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1}$ ($n \geq 1$). Let ω be the p-th root of unity ($p \geq 2$). Then the group Γ generated by ω acts on S^{2n+1} by $\omega \cdot (z_0, z_1, \ldots, z_n) = (\omega z_0, \omega z_1, \ldots, \omega z_n)$. Let the lens space be $L^{2n+1}(p) = S^{2n+1}/\Gamma$, the quotient space of S^{2n+1} by Γ. See [24, Example 3, p. 91].

Proposition 5.8 ([24, Theorem (7.9), Chapter II]) *Let p be an odd prime.*

$$H^*(L^{2n+1}(p); \mathbb{Z}/p) = \bigwedge_{\mathbb{Z}/p}(x_1) \otimes \{\mathbb{Z}/p\,[x_2]/(x_2^{n+1})\},$$

where $x_1 \in H^1(L^{2n+1}(p); \mathbb{Z}/p)$ and $x_2 = \beta_p^* x_1 \in H^2(L^{2n+1}(p); \mathbb{Z}/p)$.

Proposition 5.9 *Let p be a prime.*

(i) *If $2n+1 \neq 3, 7$, then $L^{2n+1}(p)$ is not a G-space.*
(ii) *If $2n+1 \neq 3, 7$, then $L^{2n+1}(p)$ is a NC-space.*

Proof (i) If $L^{2n+1}(p)$ is a G-space, then S^{2n+1} is a G-space [6, Theorem 2.2].
(ii) If $L^{2n+1}(p)$ is a C_k-space, then S^{2n+1} is a C_k-space by Theorem 4.10. ■

Let us recall that $L^3(p)$ is a G-space by [15, Corollary II.10], since $S^3 = \mathrm{Sp}(1)$ is a Lie group. For general $L^{2n+1}(p)$, we only know that $\pi_1(L^{2n+1}(p)) = G_1(L^{2n+1}(p))$ by [2, Theorem] or [19, Theorem A]. See also [4, Theorems II.4, II.5] and [5, Theorem 6.2]. However, for $L^3(p)$, we obtain the result using an argument similar to [15], including a proof for the fundamental group that is simpler than [2, 19] in this particular case.

Example 5.10 $L^3(p)$ is a G-space for any $p \geq 2$.

Actually, we can show the result in this way. Assume that $\pi_1(L^3(p)) = \mathbb{Z}/p$ is generated by the inclusion map $\alpha\colon S^1 \hookrightarrow L^3(p)$, which has a lift $\tilde{\alpha}\colon [0,1] \to S^3$ such that $\tilde{\alpha}(0) = 1$, $\tilde{\alpha}(1) = \xi$ and $\pi \circ \tilde{\alpha} = \alpha \circ \omega$, where $\pi\colon S^3 \to L^3(p)$ is the canonical projection taking the orbit space by the action of $\langle \xi \mid \xi^p \rangle \cong \mathbb{Z}/p$ a subgroup of a Lie group S^3, and where $\omega\colon [0,1] \to S^1$ is the standard identification map. Since S^3 is a Lie group, there is an associative unital multiplication $\mu\colon S^3 \times S^3 \to S^3$ that defines a map $\tilde{f}\colon [0,1] \times S^3 \to S^3$ by $\tilde{f} = \mu \circ (\tilde{\alpha} \times 1)$. Then $\tilde{f}$ induces a map f of orbit spaces by the action of $\mathbb{Z}/p$, since $\tilde{f}(1, \xi^i \cdot x) = \tilde{\alpha}(1) \cdot \xi^i \cdot x = \xi \cdot \xi^i \cdot x = \xi^{i+1} \cdot x = \xi^{i+1} \cdot \tilde{f}(0, x)$:

$$\begin{array}{ccccc}
[0,1] \times S^3 & \xrightarrow{\tilde{f}} & S^3 & \xleftarrow{\tilde{\alpha}} & [0,1] \\
\downarrow{\scriptstyle \omega \times \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \omega} \\
S^1 \times L^3(p) & \xrightarrow{f} & L^3(p) & \xleftarrow{\alpha} & S^1 \\
\cup & \nearrow{\scriptstyle \langle \alpha, 1_{L^3(p)} \rangle} & & & \\
S^1 \vee L^3(p), & & & &
\end{array}$$

Thus $\alpha \in G_1(L^3(p))$ and hence $G_1(L^3(p)) = \pi_1(L^3(p))$. Since the universal cover of $L^3(p)$ is S^3, which is a Lie group, we see that the projection $\pi\colon S^3 \to L^3(p)$ is a cyclic map, and hence $G_n(L^3(p)) = \pi_n(L^3(p))$ for $n \geq 2$. It follows that $L^3(p)$ is a G-space.

To examine the existence of a C_k-structure on $L^3(p)$, we need the following lemma for a space X using observations on $\Sigma\Omega X$.

Lemma 5.11 *Let X be a 0-connected CW-complex whose universal cover $\tilde{X}$ satisfies that $\Sigma\Omega\tilde{X}$ has the homotopy type of a wedge sum of spheres. Then X is a C_1-space if and only if X is a G-space.*

Proof Since $\Omega X \simeq \pi_1(X) \times \Omega\tilde{X}$, we have

$$\Sigma\Omega X \simeq \Big(\bigvee_{0 \neq \lambda \in \pi_1(X)} S^1_\lambda\Big) \vee \Sigma\Omega\tilde{X} \vee \Big(\bigvee_{0 \neq \lambda \in \pi_1(X)} S^1_\lambda \wedge \Omega\tilde{X}\Big),$$

which has the homotopy type of a wedge of spheres. Thus we have the lemma. ∎

Proposition 5.12 *$L^3(p)$ is a C_1-space for any $p \geq 2$.*

Proof By Example 5.10 and Lemma 5.11, we have the result. ■

Theorem 5.13 *$L^3(p)$ is a C_2-space if and only if $p = 2$.*

Remark When $p = 2$, the lens space $L^3(2)$ $(= RP^3 \cong \mathrm{SO}(3))$ is actually an H-space (see [12, Remark (1), p. 616]), and hence a C_k-space for any k.

Proof of Theorem 5.13 By Proposition 5.12, we know that $L^3(p)$ is a C_1-space. We also know that $L^3(2) = RP^3 = \mathrm{SO}(3)$ is a Lie group. So we are left to show that $L^3(p)$ is not a C_2-space when $p \neq 2$. If $L^3(p)$ is a C_2-space, then there is a map

$$m\colon P^2(\Omega L^3(p)) \times L^3(p) \to L^3(p)$$

whose axes are $e_2^{L^3(p)}\colon P^2(\Omega L^3(p)) \to L^3(p)$ and the identity of $L^3(p)$.

Let $L^3(p)^{(2)} = S^1 \cup e_2$ be the 2-skeleton of $L^3(p) = S^1 \cup e_2 \cup e_3$. Then there is a map $s_2\colon L^3(p)^{(2)} \to P^2(\Omega L^3(p)^{(2)}) \subset P^2(\Omega L^3(p))$ such that $e_2^{L^3(p)} \circ s_2 \sim i_2\colon L^3(p)^{(2)} \hookrightarrow L^3(p)$ is the canonical inclusion. On the other hand, we have

$$\begin{aligned} H^*(L^3(p);\mathbb{Z}/p) &\cong \bigwedge_{\mathbb{Z}/p}(x_1) \otimes \{\mathbb{Z}/p[x_2]/(x_2^2)\} \\ &\cong H^*(L^3(p)^{(2)};\mathbb{Z}/p) \oplus \mathbb{Z}/p\{x_1x_2\}, \quad \ker i_2^* = \mathbb{Z}/p\{x_1x_2\}, \end{aligned}$$

where x_i is in $H^i(L^3(p)^{(2)};\mathbb{Z}/p) \subset H^i(L^3(p);\mathbb{Z}/p)$ with a Bockstein relation $\beta_p x_1 = x_2$. Thus $(e_2^{L^3(p)})^* x_i \neq 0$ for $i = 1, 2$, since $e_2^{L^3(p)} \circ s_2 \sim i_2$.

Now let $h\colon \Sigma P^2(\Omega L^3(p)) \wedge L^3(p) \to \Sigma L^3(p)$ be the Hopf construction of the map $m\colon P^2(\Omega L^3(p)) \times L^3(p) \to L^3(p)$, and let C_h be the mapping cone of h. Then the connecting homomorphism

$$\delta\colon H^5(\Sigma P^2(\Omega L^3(p)) \wedge L^3(p);\mathbb{Z}/p) \to H^6(C_h;\mathbb{Z}/p)$$

is an isomorphism, since $H^q(\Sigma L^3(p);\mathbb{Z}/p) = 0$ for $q \geq 5$. Thus we have

$$\begin{aligned} &H^6(C_h;\mathbb{Z}/p) \cong \\ &\qquad H^4(P^2(\Omega L^3(p)) \wedge L^3(p);\mathbb{Z}/p) \supset H^2(L^3(p)^{(2)};\mathbb{Z}/p) \otimes H^2(L^3(p);\mathbb{Z}/p). \end{aligned}$$

Let $s^*\colon H^n(\Sigma X) \to H^{n-1}(X)$ be the suspension homomorphism $(n \geq 1)$. For dimensional reasons, we know that x_1 and x_2 are primitive with respect to m, and hence $s^{*-1}x_i$ lies in the image of the restriction $H^{i+1}(C_h;\mathbb{Z}/p) \to H^{i+1}(\Sigma L^3(p);\mathbb{Z}/p)$, say $y_{i+1}|_{\Sigma L^3(p)} = s^{*-1}x_i$ for $i = 1, 2$. Then by [22, Corollary 1.4(a)], we know

$$y_3^2 = \pm\delta(s^{*-1}(x_2 \otimes x_2)) \neq 0,$$

while we know that $y_3^2 = -y_3^2$ and hence $2y_3^2 = 0$. Thus we have $p = 2$. ■

Making use of the classification of GW-spaces of type (q, n, m) in [12, Theorem 1], the following result is proved.

Theorem 5.14 *Let X be a C_k-space for some $k \geq 1$ with at most three cells (other than the base point 0-cell). Then X has the homotopy type of one of the spaces in the following list.*

(i) $X = S^1, S^3, S^7$ *or their products*; *otherwise;*

(ii) *If $\pi_1(X)$ is a non-zero finite group, then $X = L^3(p, \ell)$ for an integer $p \geq 2$, where ℓ is a unit of the quotient ring $\mathbb{Z}\pi/(1 + \tau + \cdots + \tau^{p-1})$ of the group ring $\mathbb{Z}\pi$ for the group $\pi = \langle \tau \mid \tau^p = 1 \rangle \cong \mathbb{Z}/p$;*

(iii) *If $\pi_1(X) = 0$, then $X = SU(3)$ or $E_{k\omega}$ ($k \not\equiv 2 \mod 4$); in the latter case $E_{k\omega}$ is an H-space.*

Proof Since a C_k-space for some $k \geq 1$ is a T-space and hence a GW-space, we can examine the GW-spaces with up to 3 cells listed in Theorem 1 of [12]. However, CP^3 in the theorem is an NC-space by Example 5.7, and hence the result follows. ■

Remark 5.15 In view of Theorem 5.14 we see that any real, complex or quaternionic Stiefel manifold of 2-frames is an NC-space unless it is an H-space. We note that a Stiefel manifold is an H-space if and only if it is a Lie group or S^7, by [8, Theorems 1.1, 1.2] and [9, Corollary 0.6].

References

[1] J. Aguadé, *Decomposable free loop spaces*. Canad. J. Math. **39**(1987), no. 4, 938–955. doi:10.4153/CJM-1987-047-9

[2] S. A. Broughton, *The Gottlieb group of finite linear quotients of odd-dimensional spheres*. Proc. Amer. Math. Soc. **111**(1991), no. 4, 1195–1197.

[3] T. Ganea, *Lusternik-Schnirelmann category and strong category*. Illionis J. Math. **11**(1967), 417–427.

[4] D. H. Gottlieb, *A certain subgroup of the fundamental group*. Amer. J. Math. **87**(1965), 840–856. doi:10.2307/2373248

[5] ———, *Evaluation subgroups of homotopy groups*. Amer. J. Math. **91**(1969), 729–756. doi:10.2307/2373349

[6] ———, *On the construction of G-spaces and applications to homogeneous spaces*. Proc. Cambridge Philos. Soc. **68**(1970), 321–327. doi:10.1017/S0305004100046120

[7] H. B. Haslam, *G-spaces* mod *F and H-spaces* mod *F*. Duke Math. J. **38**(1971), 671–679. doi:10.1215/S0012-7094-71-03882-8

[8] J. R. Hubbuck, *Hopf structures on Stiefel manifolds*. Math. Ann. **262**(1983), no. 4, 529–547. doi:10.1007/BF01456067

[9] N. Iwase, *H-spaces with generating subspaces*. Proc. Roy. Soc. Edinburgh Sect. A **111**(1989), no. 3-4, 199–211.

[10] ———, *Ganea's conjecture on Lusternik-Schnirelmann category*. Bull. London Math. Soc. **30**(1998), no. 6, 623–634. doi:10.1112/S0024609398004548

[11] ———, *The Ganea conjecture and recent developments on Lusternik-Schnirelmann category*. Sugaku Expositions **20**(2007), no. 1, 43–63.

[12] N. Iwase, A. Kono and M. Mimura, *Generalized Whitehead spaces with few cells*. Publ. Res. Inst. Math. Sci. **28**(1992), no. 4, 615–652. doi:10.2977/prims/1195168211

[13] N. Iwase and N. Oda, *Splitting off rational parts in homotopy types*. Topology Appl. **153**(2005), no. 1, 133–140. doi:10.1016/j.topol.2005.01.027

[14] I. M. James, *On category in the sense of Lusternik-Schnirelmann*. Topology **17**(1978), no. 4, 331–348. doi:10.1016/0040-9383(78)90002-2

[15] G. E. Lang, Jr, *Evaluation subgroups of factor spaces*. Pacific J. Math. **42**(1972), 701–709.

[16] J. Milnor, *Construction of universal bundles. I, II*. Ann. Math. **63**(1956), 272–284, 430–436. doi:10.2307/1969609

[17] N. Oda, *The homotopy set of the axes of pairings*. Canad. J. Math. **17**(1990), no. 5, 856–868. doi:10.4153/CJM-1990-044-3

[18] ———, *Pairings and copairings in the category of topological spaces*. Publ. Res. Inst. Math. Sci. **28**(1992), no. 1, 83–97. doi:10.2977/prims/1195168857
[19] J. Oprea, *Finite group actions on spheres and the Gottlieb group*. J. Korean Math. Soc. **28**(1991), no. 1, 65–78.
[20] J. Siegel, *G-spaces, H-spaces and W-spaces*. Pacific J. Math. **31**(1969), 209–214.
[21] J. D. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108**(1963), 275–292, 293–312.
[22] E. Thomas, *On functional cup-products and the transgression operator*. Arch. Math. (Basel) **12**(1961), 435–444.
[23] K. Varadarajan, *Generalized Gottlieb groups*. J. Indian Math. Soc. **33**(1969), 141–164.
[24] G. W. Whitehead, *Elements of Homotopy Theory*. Graduate Texts in Mathematics 61. Springer-Verlag, New York, 1978.
[25] M. H. Woo and J.-R. Kim, *Certain subgroups of homotopy groups*. J. Korean Math. Soc. **21**(1984), no. 2, 109 – 120.
[26] M. H. Woo and Y. S. Yoon, *T-spaces by the Gottlieb groups and duality*. J. Austral. Math. Soc. Ser. A **59**(1995), no. 2, 193–203. doi:10.1017/S1446788700038593
[27] Y. S. Yoon, *Generalized Gottlieb groups and generalized Wang homomorphisms*. Sci. Math. Jpn. **55**(2002), no. 1, 139–148.
[28] ———, *H^f-spaces for maps and their duals*. J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. **14**(2007), no. 4, 289–306.
[29] ———, *Lifting T-structures and their duals*. J. Chungcheong Math. Soc. **20**(2007), 245–259.

Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan
e-mail: iwase@math.kyushu-u.ac.jp

Department of Mathematics, Okayama University, Okayama 700-8530, Japan
e-mail: mimura@math.okayama-u.ac.jp

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
e-mail: odanobu@cis.fukuoka-u.ac.jp

Department of Mathematics Education, Hannam University, Daejeon 306-791, Korea
e-mail: yoon@hannam.ac.kr