ON LUSTERNIK-SCHNIRELMANN CATEGORY OF SO(10)

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Abstract. Let G be a compact connected Lie group and $p : E \to \Sigma A$ be a principal G-bundle with a characteristic map $\alpha : A \to G$, where $A = \Sigma A_0$ for some A_0. Let $\{K_i{\to}F_{i-1}{\hookrightarrow}F_i \mid 1 \le i \le m\}$ with $F_0{=}\{*\}$, $F_1{=}\Sigma K_1$ and $F_m{\simeq}G$ be a cone-decomposition of G of length m and $F'_1 = \Sigma K'_1 \subset F_1$ with $K'_1 \subset K_1$ which satisfy $F_iF'_1 \subset F_{i+1}$ up to homotopy for all i. Then we have $\mathrm{cat}(E) \le m{+}1$, under some suitable conditions, which is used to determine $\mathrm{cat}(\mathbf{SO}(10))$. A similar result is obtained by Kono and the first author [9] to determine $\mathrm{cat}(\mathbf{Spin}(9))$, while the result in [9] can not assert $\mathrm{cat}(E) \le m{+}1$.

1. Introduction

Throughout the paper, we work in the homotopy category of based CW-complexes, and often identify a map with its homotopy class.

The Lusternik-Schnirelmann category of a connected space X, denoted by $\mathrm{cat}(X)$, is the least integer n such that there is an open covering $\{U_i \mid 0 \le i \le n\}$ of X with each U_i contractible in X. If no such integer exists, we write $\mathrm{cat}(X) = \infty$. Let R be a commutative ring with unit. The cup-length of X w.r.t. R, denoted by $\mathrm{cup}(X;R)$, is the supremum of all non-negative integers k such that there is a non-zero k-fold cup product in the ordinary reduced cohomology $\tilde{H}^*(X;R)$.

In 1967, Ganea introduced in [3] a strong category $\mathrm{Cat}(X)$ by modifying Fox's strong category (see Fox [2]), which is characterized as follows: for a connected space X, $\mathrm{Cat}(X)$ is 0 if X is contractible and, otherwise, is equal to the smallest integer n such that there is a series of cofibre sequences $\{K_i \to F_{i-1} \hookrightarrow F_i \mid 1 \le i \le m\}$ with $F_0 = \{*\}$ and $F_m \simeq X$ (a cone-decomposition of length m). $\mathrm{Cat}(X)$ is often called the cone-length of X. The following theorem is well-known.

Theorem 1.1 (Ganea [3])**.** $\mathrm{cup}(X;R) \le \mathrm{cat}(X) \le \mathrm{Cat}(X)$.

In 1968, Berstein and Hilton [1] gave a criterion for $\mathrm{cat}(C_f) = 2$ in terms of their Hopf invariant $H_1(f) \in [\Sigma X, \Omega\Sigma Y * \Omega\Sigma Y]$ for a map $f : \Sigma X \to \Sigma Y$,

2010 *Mathematics Subject Classification.* Primary 55M30; Secondary 55Q25, 55R10, 57T10, 57T15.

Key words and phrases. Lusternik-Schnirelmann category, special orthogonal groups, Hopf invariant, principal bundle.

where $A{*}B$ denotes the join of spaces A and B. In addition, its higher version H_m is used to disprove the Ganea conjecture (see Iwase [6, 8]).

We summarize here known L-S categories of special orthogonal groups: since $\mathbf{SO}(2) = S^1$, $\mathbf{SO}(3) = \mathbb{R}P^3$ and $\mathbf{SO}(4) = \mathbb{R}P^3{\times}S^3$, we know

$$\mathrm{cat}(\mathbf{SO}(2)) = 1, \quad \mathrm{cat}(\mathbf{SO}(3)) = 3 \quad \text{and} \quad \mathrm{cat}(\mathbf{SO}(4)) = 4.$$

In 1999, James and Singhof [12] gave the first non-trivial result.

$$\mathrm{cat}(\mathbf{SO}(5)) = 8.$$

In 2005, Mimura, Nishimoto and the first author [11] gave an alternative proof of $\mathrm{cat}(\mathbf{SO}(5)) = 8$ and determine $\mathrm{cat}(\mathbf{SO}(n))$ up to $n{=}9$ as follows.

$$\mathrm{cat}(\mathbf{SO}(6)) = 9,\ \mathrm{cat}(\mathbf{SO}(7)) = 11,\ \mathrm{cat}(\mathbf{SO}(8)) = 12 \ \text{and}\ \mathrm{cat}(\mathbf{SO}(9)) = 20.$$

Let $G \hookrightarrow E \to \Sigma A$ be a principal bundle with a characteristic map $\alpha : A \to G$, where A is a suspension space and G is a connected compact Lie group with a cone-decomposition of length m, i.e., there is a series of cofibre sequences $\{K_i \to F_{i-1} \hookrightarrow F_i \,|\, 1 \le i \le m\}$ with $F_0 = \{*\}$, $F_1 \simeq \Sigma K_1$ and $F_m \simeq G$. Then the multiplication of G is, up to homotopy, a map $\mu : F_m{\times}F_m \to F_m$, since $G \simeq F_m$. The main result of this paper is as follows.

Theorem 1.2. *Let $F'_1 = \Sigma K'_1$, where K'_1 is a connected subspace of K_1 so that F'_1 is simply-connected, and let $\mu|_{F_i \times F'_1} : F_i \times F'_1 \to F_m$ be compressible into $F_{i+1} \subset F_m$ as $\mu_{i,1} : F_i \times F'_1 \to F_{i+1}$, $1{\le}i{<}m$, such that $\mu_{i,1}|_{F_{i-1}\times F'_1} \sim \mu_{i-1,1}$ in F_{i+1}. Then the following three conditions imply* $\mathrm{cat}(E) \le m{+}1$.

(1) α is compressible into F'_1,
(2) $H_1(\alpha) = 0$ in $[A, \Omega F'_1{*}\Omega F'_1]$,
(3) $K_m = S^{\ell-1}$ with $m{\ge}3$ and $\ell{\ge}3$.

Remark. Under the conditions in Theorem 1.2, [9, Theorem 0.8] does *not* imply $\mathrm{cat}(E) \le m{+}1$, but only does $\mathrm{cat}(E) \le m{+}2$, since its key lemma [9, Lemma 1.1] can not properly manage the case when $\mathrm{im}\,\alpha \subset F_1$.

Theorem 1.2 yields the following result on L-S category of $\mathbf{SO}(10)$.

Theorem 5.1. $\mathrm{cat}(\mathbf{SO}(10)) = \mathrm{cup}(\mathbf{SO}(10); \mathbb{F}_2) = 21$.

All these results on $\mathrm{cat}(\mathbf{SO}(n))$ with $n{\le}10$ support the "folk conjecture".

Conjecture 1. $\mathrm{cat}(\mathbf{SO}(n)) = \mathrm{cup}(\mathbf{SO}(n); \mathbb{F}_2)$.

Let us explain the method we employ in this paper. To study L-S category, we must understand Ganea's criterion of L-S category as a basic idea, given in terms of a fibre-cofibre construction in [3]: let X be a connected space. Then there is a fibre sequence $F_nX \hookrightarrow G_nX \to X$, natural with respect to X, such that $\mathrm{cat}(X) \leq n$ if and only if the fibration $G_nX \to X$ has a cross-section.

However, four years before [3], a more understandable description of the fibre sequence $F_n(X) \hookrightarrow G_n(X) \to X$ was already published by Stasheff [15]: following [6, 7, 8], we may replace the inclusion $F_nX \hookrightarrow G_nX$ with the fibration $p^{\Omega X}_n : E^{n+1}\Omega X \to P^n\Omega X$ associated with the A_∞ structure of ΩX the based loop space of X in the sense of Stasheff, where $E^{n+1}\Omega X$ has the homotopy type of $(\Omega X)^{*(n+1)}$ the $n{+}1$-fold join of ΩX and $P^n\Omega X$ satisfies $P^0\Omega X = *$, $P^1\Omega X = \Sigma\Omega X$ and $P^\infty\Omega X \simeq X$. Let $\iota^{\Omega X}_{m,n} : P^m\Omega X \hookrightarrow P^n\Omega X$ be the canonical inclusion, for $m \leq n$, and $e^X_\infty : P^\infty\Omega X \simeq X$ be the natural equivalence. Then the fibration $G_nX \to X$ may be replaced with the map $e^X_n = e^X_\infty \circ \iota^{\Omega X}_{n,\infty} : P^n\Omega X \to X$, where $e^X_1 : \Sigma\Omega X \to X$ equals the evaluation.

Thus, we may restate Ganea's criterion as below: let X be a connected space. Then $\mathrm{cat}(X) \leq n$ if and only if $e^X_n : P^n\Omega X \to X$ has a right homotopy inverse. It is the reason why we use A_∞-structures to determine L-S category.

In this paper, instead of using [9, Lemma 1.1], we show Proposition 2.4, Lemma 3.3 and Lemma 4.4. It is a key process to obtain Theorem 1.2. In Sections 2 and 3, we construct a structure map associated to a given cone-decomposition. In Section 4, we introduce a map $\hat{\lambda}$ from $\hat{F}_{m+1} = P^m_m \times \Sigma\Omega F'_1$ to $P^{m+1}\Omega F_m$, which is the main tool to construct a complex D of $\mathrm{Cat}(D) \leq m{+}1$ dominating E. Finally in Section 5, we prove Theorem 5.1.

2. Structure Map Associated With Cone-Decomposition

In this section, we generalize the following well-known fact to a proposition for filtered spaces and maps.

Fact 2.1. *Let $K \xrightarrow{a} A \hookrightarrow C(a)$, $L \xrightarrow{b} B \hookrightarrow C(b)$ be cofibre sequences with canonical co-pairings $\nu : C(a) \to C(a) \vee \Sigma K$ and $\hat{\nu} : C(b) \to C(b) \vee \Sigma L$. If there are maps $f : A \to B$ and $f^0 : K \to L$ such that $f \circ a = b \circ f^0$, then they induce a map $f' : C(a) \to C(b)$ satisfying $(f' \vee \Sigma f^0) \circ \nu = \hat{\nu} \circ f'$.*

Definition 2.2. A space X with a series of subspaces $\{X_n; n \geq 0\}$,

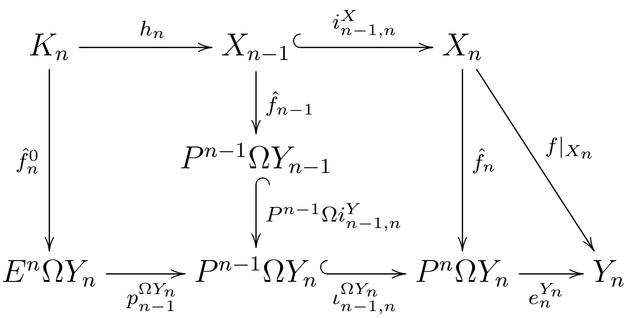
$$\{*\} = X_0 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X,$$

is called a space filtered by $\{X_n; n \geq 0\}$ and denoted by $(X, \{X_n\})$. We also denote by $i^X_{m,n} : X_m \hookrightarrow X_n$, $m < n$ the canonical inclusion.

Definition 2.3. Let X and Y be spaces filtered by $\{X_n\}$ and $\{Y_n\}$, respectively. A map $f : X \to Y$ is a filtered map if $f(X_n) \subset Y_n$ for all n.

Proposition 2.4. *Let X and Y be filtered by $\{X_n\}$ and $\{Y_n\}$, respectively, and $f : X \to Y$ be a filtered map. If $\{X_n\}$ is a cone-decomposition of X, i.e, there is a series of cofibre sequences $\{K_n \xrightarrow{h_n} X_{n-1} \xhookrightarrow{i^X_{n-1,n}} X_n \,|\, n \geq 1\}$ with $X_0 = *$, then there exist families of maps $\{\hat{f}_n : X_n \to P^n\Omega Y_n \,|\, n \geq 0\}$ and $\{\hat{f}^0_n : K_n \to E^n\Omega Y_n \,|\, n \geq 0\}$ such that they satisfy two conditions as follows.*

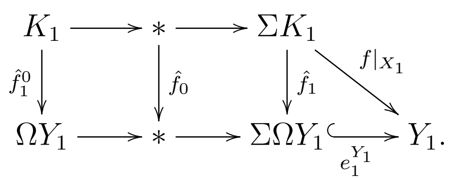
(1) The following diagram is commutative.



(2) We denote by $f'_n = (P^{n-1}\Omega i^Y_{n-1,n} \circ \hat{f}_{n-1}) \cup C(\hat{f}^0_n) : X_n \to P^n\Omega Y_n$ the induced map from the commutativity of the left square in (1). Then the middle square in (1) with $\hat{f}_n$ replaced with f'_n is commutative. The difference of $\hat{f}_n$ and f'_n is given by a map $\delta^f_n : \Sigma K_n \to P^{n-1}\Omega Y_n$ composed with the inclusion $\iota^{\Omega Y_n}_{n-1,n} : P^{n-1}\Omega Y_n \hookrightarrow P^n\Omega Y_n$, $n \geq 1$.

Proof. First of all, we put $\hat{f}_0 = *$ the trivial map.

Next, we show the proposition by induction on $n \geq 1$. When $n = 1$, we put $\hat{f}^0_1 = \mathrm{ad}(f|_{X_1})$ and $\hat{f}_1 = \Sigma\,\mathrm{ad}(f|_{X_1}) = f'_1$ to obtain the following commutative diagram:



Then (1) is clear and (2) is trivial in this case.

When $n = k > 1$, suppose we have already obtained $\{\hat{f}_i\}$ and $\{\hat{f}^0_i\}$ for $i < k$, which satisfies the conditions (1) and (2).

Firstly, we define $\hat{f}^0_k : K_k \to E^k\Omega Y_k$ as follows: the homotopy class of a map $P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \circ h_k : K_k \to P^{k-1}\Omega Y_k$ can be described as

$$h_{k*}(P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1}) \in [K_k, Y_k] \text{ with } P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \in [X_{k-1}, Y_k]$$

in the following ladder of exact sequences induced from a fibre sequence $E^k\Omega Y_k \to P^{k-1}\Omega Y_k \to Y_k$:

$$\begin{array}{ccccc}
[X_{k-1}, E^k\Omega Y_k] & \xrightarrow{p^{\Omega Y_k}_{k-1*}} & [X_{k-1}, P^{k-1}\Omega Y_k] & \xrightarrow{e^{Y_k}_{k-1*}} & [X_{k-1}, Y_k] \\
\downarrow{\scriptstyle h_k^*} & & \downarrow{\scriptstyle h_k^*} & & \downarrow{\scriptstyle h_k^*} \\
[K_k, E^k\Omega Y_k] & \xrightarrow{p^{\Omega Y_k}_{k-1*}} & [K_k, P^{k-1}\Omega Y_k] & \xrightarrow{e^{Y_k}_{k-1*}} & [K_k, Y_k].
\end{array}$$

Since we know that the naturality of e^Z_{k-1} at Z implies $e^{Y_k}_{k-1}\circ P^{k-1}\Omega i^Y_{k-1,k} = i^Y_{k-1,k}\circ e^{Y_{k-1}}_{k-1}$, that the induction hypothesis implies $e^{Y_{k-1}}_{k-1}\circ\hat{f}_{k-1} = f|_{X_{k-1}}$ and that the naturality of $i^Z_{k-1,k}$ at Z implies $i^Y_{k-1,k}\circ f|_{X_{k-1}} = f|_{X_k}\circ i^X_{k-1,k}$, we obtain $e^{Y_k}_{k-1*}(P^{k-1}\Omega i^Y_{k-1,k}\circ\hat{f}_{k-1}) = i^Y_{k-1,k}\circ e^{Y_{k-1}}_{k-1}\circ\hat{f}_{k-1} = f|_{X_k}\circ i^X_{k-1,k} \in [X_{k-1}, Y_k]$. On the other hand, since $K_k \to X_{k-1} \hookrightarrow X_k$ is a cofibre sequence, we obtain

$$e^{Y_k}_{k-1*}(h_k^*(P^{k-1}\Omega i^Y_{k-1,k}\circ\hat{f}_{k-1})) = [f|_{X_k}\circ i^X_{k-1,k}\circ h_k] = 0.$$

Thus we have $e^{Y_k}_{k-1*}(P^{k-1}\Omega i^Y_{k-1,k}\circ\hat{f}_{k-1}\circ h_k) = 0$ and there exists a map $\hat{f}^0_k : K_k \to E^k\Omega Y_k$ such that $p^{\Omega Y_k}_{k-1*}(\hat{f}^0_k) = P^{k-1}\Omega i^Y_{k-1,k}\circ\hat{f}_{k-1}\circ h_k$, which implies the commutativity of the left square in (1).

Secondly, let $f'_k : X_k \to P^k\Omega Y_k$ be the map induced from the commutativity of the left square in (1). By the induction hypothesis, we have

$$\begin{aligned}
(i^X_{k-1,k})^*(e^{Y_k}_k\circ f'_k) &= [e^{Y_k}_k\circ f'_k\circ i^X_{k-1,k}] = [e^{Y_k}_k\circ \iota^{\Omega Y_k}_{k-1,k}\circ P^{k-1}\Omega i^Y_{k-1,k}\circ\hat{f}_{k-1}] \\
&= [i^Y_{k-1,k}\circ e^{Y_{k-1}}_{k-1}\circ\hat{f}_{k-1}] = [i^Y_{k-1,k}\circ f|_{X_{k-1}}] = [f|_{X_k}\circ i^X_{k-1,k}] = (i^X_{k-1,k})^*(f|_{X_k}).
\end{aligned}$$

By a standard argument of homotopy theory on a cofibre sequence $K_k \to X_{k-1} \hookrightarrow X_k$ (see Hilton [5] or Oda [13]), there is a map $\delta^{f,0}_k : \Sigma K_k \to Y_k$ such that

$$f|_{X_k} = \nabla_{Y_k}\circ(e^{Y_k}_k\circ f'_k \vee \delta^{f,0}_k)\circ\nu_k,$$

where $\nabla_Y : Y \vee Y \to Y$ denotes the folding map for a space Y and $\nu_k : X_k \to X_k \vee \Sigma K_k$ denotes the canonical co-pairing.

Let $\delta^f_k = \iota^{\Omega Y_k}_{1,k-1}\circ\Sigma\,\mathrm{ad}(\delta^{f,0}_k) : \Sigma K_k \to \Sigma\Omega Y_k \hookrightarrow P^{k-1}\Omega Y_k$. Since $e^{Y_k}_1 = e^{Y_k}_{k-1}\circ\iota^{\Omega Y_k}_{1,k-1}$, we have $\delta^{f,0}_k = e^{Y_k}_1\circ\Sigma\mathrm{ad}(\delta^{f,0}_k) = e^{Y_k}_{k-1}\circ\delta^f_k$. Hence, we obtain $\hat{f}_k = \nabla_{P^k\Omega Y_k}\circ(f'_k \vee \iota^{\Omega Y_k}_{k-1,k}\circ\delta^f_k)\circ\nu_k$ satisfies the condition (2).

Thirdly, by using the above homotopy relations, we obtain the following.

$$\begin{aligned}
f|_{X_k} &= \nabla_{Y_k}\circ(e^{Y_k}_k\circ f'_k \vee e^{Y_k}_{k-1}\circ\delta^f_k)\circ\nu_k \\
&= e^{Y_k}_k\circ\nabla_{P^k\Omega Y_k}\circ(f'_k \vee \iota^{\Omega Y_k}_{k-1,k}\circ\delta^f_k)\circ\nu_k = e^{Y_k}_k\circ\hat{f}_k.
\end{aligned}$$

This implies the commutativity of the right triangle in (1).

Finally, since ν_k is a co-pairing, we have

$$pr_1\circ\nu_k\circ i^X_{k-1,k} = 1_{X_k}\circ i^X_{k-1,k} = i^X_{k-1,k} \quad\text{and}\quad pr_2\circ\nu_k\circ i^X_{k-1,k} = q\circ i^X_{k-1,k} = *,$$

where $pr_1 : X_k \vee \Sigma K_k \to X_k$ and $pr_2 : X_k \vee \Sigma K_k \to \Sigma K_k$ are the first and second projections, respectively. Then, we obtain the equation

$$\begin{aligned} \hat{f}_k \circ i^X_{k-1,k} &= \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta^f_k) \circ \nu_k \circ i^X_{k-1,k} \\ &= f'_k \circ i^X_{k-1,k} = \iota^{\Omega Y_k}_{k-1,k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1}, \end{aligned}$$

which implies the commutativity of the middle square in (1). This completes the induction step for $n = k$, and we obtain the proposition for all n. □

Corollary 2.4.1. *Let $\hat{\nu}_n : P^n\Omega Y_n \to P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n$ be the canonical co-pairing. If K_n is a co-H-space, then the following diagram is commutative.*

$$\begin{array}{ccc} X_n & \xrightarrow{\nu_n} & X_n \vee \Sigma K_n \\ \Big\downarrow{\scriptstyle \hat{f}_n} & & \Big\downarrow{\scriptstyle \hat{f}_n \vee \Sigma \hat{f}^0_n} \\ P^n\Omega Y_n & \xrightarrow{\hat{\nu}_n} & P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n. \end{array}$$

Proof. Let P and E denotes $P^n\Omega Y_n$ and $E^n\Omega Y_n$, respectively. By Proposition 2.4 (2), the difference of $\hat{f}_n$ and f'_n is given by $\iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n$, and hence

$$\begin{aligned} (\hat{f}_n \vee \Sigma \hat{f}^0_n) \circ \nu_n &= \{(\nabla_P \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n) \circ \nu_n) \vee \Sigma \hat{f}^0_n\} \circ \nu_n \\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma \hat{f}^0_n) \circ (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n. \end{aligned}$$

Since K_n is a co-H-space, we have the following homotopy relations.

$$\upsilon_n = T \circ \upsilon_n \quad \text{and} \quad (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n = (1_{X_n} \vee \upsilon_n) \circ \nu_n,$$

where $\upsilon_n : \Sigma K_n \to \Sigma K_n \vee \Sigma K_n$ is the co-multiplication and $T : \Sigma K_n \vee \Sigma K_n \to \Sigma K_n \vee \Sigma K_n$ is a switching map. So we can proceed as follows:

$$\begin{aligned} (\hat{f}_n \vee \Sigma \hat{f}^0_n) \circ \nu_n &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma \hat{f}^0_n) \circ (1_{X_n} \vee \upsilon_n) \circ \nu_n \\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee (\iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma \hat{f}^0_n)) \circ (1_{X_n} \vee T \circ \upsilon_n) \circ \nu_n \\ &= (\nabla_P \vee 1_{\Sigma E}) \circ \{f'_n \vee T' \circ (\Sigma \hat{f}^0_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n)\} \circ (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n \\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (1_P \vee T') \circ \{(f'_n \vee \Sigma \hat{f}^0_n) \circ \nu_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n, \end{aligned}$$

where $T' : \Sigma E \vee P \to P \vee \Sigma E$ is a switching map. Then we can easily see that $(\nabla_P \vee 1_{\Sigma E}) \circ (1_P \vee T') = \nabla_{P \vee \Sigma E} \circ \mathrm{in}_{\Sigma E}$, where, for any space Y, we denote by $\mathrm{in}_{\Sigma E} : Y \hookrightarrow Y \vee \Sigma E$ the first inclusion. So we proceed as follows.

$$\begin{aligned} (\hat{f}_n \vee \Sigma \hat{f}^0_n) \circ \nu_n &= \nabla_{P \vee \Sigma E} \circ \mathrm{in}_{\Sigma E} \circ \{(f'_n \vee \Sigma \hat{f}^0_n) \circ \nu_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n \\ &= \nabla_{P \vee \Sigma E} \circ \{(f'_n \vee \Sigma \hat{f}^0_n) \circ \nu_n \vee \mathrm{in}_{\Sigma E} \circ \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n. \end{aligned}$$

Here, since the co-pairing $\hat{\nu}_n$ is associated to the cofibre sequence $P^{n-1}\Omega Y_n \xhookrightarrow{\iota^{\Omega Y_n}_{n-1,n}} P^n\Omega Y_n \longrightarrow \Sigma E^n\Omega Y_n$, we have the following equation up to homotopy:

$$\hat{\nu}_n \circ \iota^{\Omega Y_n}_{n-1,n} = \mathrm{in}_{\Sigma E} \circ \iota^{\Omega Y_n}_{n-1,n} : P^{n-1}\Omega Y_n \hookrightarrow P^n\Omega Y_n \hookrightarrow P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n.$$

Then by Theorem 2.1, we proceed further as follows:

$$\begin{aligned}(\hat{f}_n \vee \Sigma\hat{f}^0_n)\circ\nu_n &= \nabla_{P\vee\Sigma E}\circ\{(f'_n \vee \Sigma\hat{f}^0_n)\circ\nu_n \vee \hat{\nu}_n\circ\iota^{\Omega Y_n}_{n-1,n}\circ\delta^f_n\}\circ\nu_n\\ &= \nabla_{P\vee\Sigma E}\circ(\hat{\nu}_n\circ f'_n \vee \hat{\nu}_n\circ\iota^{\Omega Y_n}_{k-1,k}\circ\delta^f_n)\circ\nu_n\\ &= \hat{\nu}_n\circ\nabla_P\circ(f'_n \vee \iota^{\Omega Y_n}_{n-1,n}\circ\delta^f_n)\circ\nu_n = \hat{\nu}_n\circ f_n.\end{aligned}$$

It completes the proof of the corollary. □

3. Cone-Decomposition Associated with Projective Spaces

Let G be a compact Lie group of dimension ℓ with a cone-decomposition of length m, that is, there is a series of cofibre sequences

$$\{K_i \xrightarrow{h_i} F_{i-1} \hookrightarrow F_i \mid 1\leq i\leq m\} \tag{3.1}$$

with $F_0 = \{*\}$ and $F_m \simeq G$. We also denote by $i^F_{i-1,i} : F_{i-1} \hookrightarrow F_i$ the canonical inclusion and by $q^F_{i-1,i} : F_i \to F_i/F_{i-1} = \Sigma K_i$ its successive quotient.

Lemma 3.1. *If $K_m = S^{\ell-1}$ with $m\geq 3$ and $\ell\geq 3$, then we obtain*

(1) $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is an ℓ-connected pair.

(2) There exists an ℓ-connected map $\hat{\phi}_S : P^m_m = P^m\Omega F_{m-1} \cup CS^{\ell-1} \to P^m\Omega F_m$ extending the inclusion $P^m\Omega F_{m-1} \hookrightarrow P^m\Omega F_m$.

Proof. Let $q_E : \mathfrak{F}_E \to E^m\Omega F_{m-1}$, $q_P : \mathfrak{F}_P \to P^{m-1}\Omega F_{m-1}$ and $q_F : \mathfrak{F}_F \to F_{m-1}$ be homotopy fibres of inclusion maps $E^m\Omega i^F_{m-1,m}$, $P^{m-1}\Omega i^F_{m-1,m}$ and $i^F_{m-1,m}$, respectively, which fit in with the following commutative diagram of fibre sequences. Thus we obtain a fibre sequence $\mathfrak{F}_E \to \mathfrak{F}_P \to \mathfrak{F}_F$:

$$\begin{array}{ccccc} \mathfrak{F}_E & \longrightarrow & \mathfrak{F}_P & \longrightarrow & \mathfrak{F}_F \\ \downarrow{\scriptstyle q_E} & & \downarrow{\scriptstyle q_P} & & \downarrow{\scriptstyle q_F} \\ E^m\Omega F_{m-1} & \xrightarrow{p^{\Omega F_{m-1}}_{m-1}} & P^{m-1}\Omega F_{m-1} & \xrightarrow{e^{F_{m-1}}_{m-1}} & F_{m-1} \\ \downarrow{\scriptstyle E^m\Omega i^F_{m-1,m}} & & \downarrow{\scriptstyle P^{m-1}\Omega i^F_{m-1,m}} & & \downarrow{\scriptstyle i^F_{m-1,m}} \\ E^m\Omega F_m & \xrightarrow{p^{\Omega F_m}_{m-1}} & P^{m-1}\Omega F_m & \xrightarrow{e^{F_m}_{m-1}} & F_m. \end{array}$$

Firstly, since the pair (F_m, F_{m-1}) is $(\ell-1)$-connected, $(\Omega F_m, \Omega F_{m-1})$ is $(\ell-2)$-connected and $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is $(\ell+m-3)$-connected. Therefore, $\mathfrak{F}_F$ is $(\ell-2)$-connected and $\mathfrak{F}_E$ is $(\ell+m-4)$-connected. We remark that $\mathfrak{F}_E$ is at least $(\ell-1)$-connected, since $m \geq 3$, Then, by using the homotopy exact sequence for the fibre sequence $\mathfrak{F}_E \to \mathfrak{F}_P \to \mathfrak{F}_F$, we obtain

$$\pi_k(\mathfrak{F}_P) \cong \pi_k(\mathfrak{F}_F), \quad k \leq \ell-1,$$

and hence $\mathfrak{F}_P$ is $(\ell-2)$-connected. Thus $\mathfrak{F}_P$ is 1-connected, since $\ell\geq 3$. By a general version of Blakers-Massey Theorem (see [4, Corollary 16.27], for example) and the hypothesis that $K_m = S^{\ell-1}$, it follows that

$$\pi_{\ell-1}(\mathfrak{F}_P) \cong \pi_{\ell-1}(\mathfrak{F}_F) \cong \pi_\ell(F_m, F_{m-1}) \cong \pi_\ell(\Sigma K_m) \cong \pi_\ell(S^\ell) \cong \mathbb{Z},$$

Thus, $\mathfrak{F}_P$ has the following homology decomposition, up to homotopy.

$$\mathfrak{F}_P = (S^{\ell-1} \vee S^\ell \vee \cdots \vee S^\ell) \ \cup\ (\text{cells in dimension} \geq \ell{+}1).$$

Secondly, $P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ is described as the homotopy pushout of $q_P : \mathfrak{F}_P \to P^{m-1}\Omega F_{m-1}$ and the trivial map $* : \mathfrak{F}_P \to \{*\}$. Then we obtain

$$\text{(3.2)}\qquad \begin{array}{ccc} P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P & \longrightarrow & \begin{array}{c} P^{m-1}\Omega F_{m-1}{\times}P^{m-1}\Omega F_m \\ \cup P^{m-1}\Omega F_m{\times}\{*\} \end{array} \\ {\scriptstyle \phi_P}\big\downarrow & \text{HPB} & \big\downarrow \\ P^{m-1}\Omega F_m & \xrightarrow[\Delta]{} & P^{m-1}\Omega F_m{\times}P^{m-1}\Omega F_m \end{array}$$

(see [6, Lemma 2.1], for example, with $(X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$, $(Y, B) = (P^{m-1}\Omega F_m, \{*\})$ and $Z = P^{m-1}\Omega F_m$), where we denote by Δ the diagonal map. Thus there is a map $\phi_P : P^{m-1}\Omega F_{m-1} \cup_{q_P} C(\mathfrak{F}) \to P^{m-1}\Omega F_m$ as the left down arrow in the diagram (3.2). On the other hand, by the proof of [6, Lemma 2.1], the subspace $P^{m-1}\Omega F_{m-1} \subset P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ can be described as the pull-back of Δ above and the inclusion map

$$P^{m-1}\Omega i^F_{m-1,m}{\times}1 : P^{m-1}\Omega F_{m-1}{\times}P^{m-1}\Omega F_m \hookrightarrow P^{m-1}\Omega F_{m-1}{\times}P^{m-1}\Omega F_m,$$

and hence we obtain

$$\phi_P|_{P^{m-1}\Omega F_{m-1}} = P^{m-1}\Omega i^F_{m-1,m} : P^{m-1}\Omega F_{m-1} \hookrightarrow P^{m-1}\Omega F_m.$$

Thirdly, the homotopy fibre $\mathfrak{F}^0_P$ of ϕ_P is the homotopy pullback of the inclusion $P^{m-1}\Omega F_{m-1}{\times}P^{m-1}\Omega F_m{\cup}P^{m-1}\Omega F_m{\times}\{*\} \hookrightarrow P^{m-1}\Omega F_m{\times}P^{m-1}\Omega F_m$ and the trivial map $\{*\} \to P^{m-1}\Omega F_m{\times}P^{m-1}\Omega F_m$. Then we obtain

$$\begin{array}{ccc} \mathfrak{F}_P{\times}\Omega P^{m-1}\Omega F_m & \xrightarrow{\text{proj}_2} & P^{m-1}\Omega F_{m-1} \\ {\scriptstyle \text{proj}_1}\big\downarrow & \text{HPO} & \big\downarrow \\ \mathfrak{F}_P & \longrightarrow & \mathfrak{F}^0_P \end{array}$$

(see [6, Lemma 2.1], for example, with $(X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$, $(Y, B) = (P^{m-1}\Omega F_m, \{*\})$ and $Z = \{*\}$). Hence $\mathfrak{F}^0_P$ has the homotopy type of the join $\mathfrak{F}_P{*}\Omega P^{m-1}\Omega F_m$ which is $(\ell{-}1)$-connected. Thus ϕ_P is ℓ-connected.

Finally, let $q_S = q_P|_{S^{\ell-1}} : S^{\ell-1} \to P^{m-1}\Omega F_{m-1}$. Then the inclusion $j : P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1} \hookrightarrow P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ is ℓ-connected, since

$$P^{m-1}\Omega F_{m-1}{\cup_{q_P}}C\mathfrak{F}_P = P^{m-1}\Omega F_{m-1}{\cup_{q_S}}CS^{\ell-1}{\cup}(\text{cells in dimension} \geq \ell{+}1).$$

Then the composition $\phi_S = \phi_P \circ j : (P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1}, P^{m-1}\Omega F_{m-1}) \hookrightarrow (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$ of ℓ-connected maps is again ℓ-connected.

Since $m \geq 3$, the pair $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is ℓ-connected, which implies (1). Thus, the inclusion $P^{m-1}\Omega F_m \cup C(E^m\Omega F_{m-1}) \hookrightarrow P^{m-1}\Omega F_m \cup C(E^m\Omega F_m)$ is ℓ-connected, and we obtain an ℓ-connected map

$$\begin{aligned}\hat{\phi}_S : P^m\Omega F_{m-1} \cup CS^{\ell-1} &= P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1} \cup_{p_{m-1}^{\Omega F_{m-1}}} C(E^m\Omega F_{m-1}) \\ &\to P^{m-1}\Omega F_m \cup C(E^m\Omega F_{m-1}) \hookrightarrow P^{m-1}\Omega F_m \cup C(E^m\Omega F_m) = P^m\Omega F_m,\end{aligned}$$

which implies (2). It completes the proof of Lemma 3.1. □

From now on, we assume $K_m = S^{\ell-1}$ with $m \geq 3$ and $\ell \geq 3$. Thus, by Lemma 3.1, we may assume that $P^m_m = P^m\Omega F_{m-1} \cup CS^{\ell-1} \subset P^m\Omega F_m$ such that $(P^m\Omega F_m, P^m_m)$ is ℓ-connected. In this section, we define cone-decompositions of $F_m \times F'_1$, P^m_m and $P^m_m \times \Sigma\Omega F'_1$.

Firstly, we give a cone-decomposition of $F_m \times F'_1$ of length $m+1$ as follows.

$$\{K^{m,1}_i \xrightarrow{w^{m,1}_i} F^{m,1}_{i-1} \hookrightarrow F^{m,1}_i \mid 1 \leq i \leq m+1\} \quad \text{with} \quad F^{m,1}_{m+1} = F_m \times F'_1, \tag{3.3}$$

where $K^{m,1}_i$, $F^{m,1}_{i-1}$ and $w^{m,1}_i$ $(1 \leq i \leq m+1)$ are defined by

$$K^{m,1}_1 = K_1 \vee K'_1, \qquad F^{m,1}_0 = \{*\}, \qquad w^{m,1}_1 = * : K^{m,1}_1 \to F^{m,1}_0,$$

$$\begin{cases} K^{m,1}_i = K_i \vee (K_{i-1} * K'_1), \qquad F^{m,1}_{i-1} = F_{i-1} \times \{*\} \cup F_{i-2} \times F'_1, \\ w^{m,1}_i|_{K_i} = \mathrm{incl} \circ (h_i \times *) : K_i \to F_{i-1} = F_{i-1} \times \{*\} \subset F^{m,1}_{i-1}, \\ w^{m,1}_i|_{K_{i-1}*K'_1} = [\chi_{i-1}, \Sigma\, 1_{K'_1}]^r \\ \qquad : K_{i-1} * K'_1 \to F_{i-1} \times \{*\} \cup F_{i-2} \times \Sigma K'_1 = F^{m,1}_{i-1}, \end{cases} \quad i \geq 2,$$

in which $K_{m+1} = \{*\}$, incl is the canonical inclusion and $[\chi_i, \Sigma\, 1_{K'_1}]^r$ is the relative Whitehead product of the characteristic map $\chi_i : (CK_i, K_i) \to (F_i, F_{i-1})$ and the suspension of the identity map $\Sigma\, 1_{K'_1} : \Sigma K'_1 \to \Sigma K'_1$.

Secondly, a cone-decomposition of P^m_m of length m is given as follows.

$$\begin{cases} \Omega F_{m-1} \to \{*\} \hookrightarrow \Sigma\Omega F_{m-1}, \\ \qquad \vdots \\ E^i\Omega F_{m-1} \to P^{i-1}\Omega F_{m-1} \hookrightarrow P^i\Omega F_{m-1}, \qquad 1 \leq i < m, \\ \qquad \vdots \\ E^m\Omega F_{m-1} \vee K_m \to P^{m-1}\Omega F_{m-1} \hookrightarrow P^m_m. \end{cases}$$

Finally, a cone-decomposition of $P^m_m \times \Sigma\Omega F'_1$ of length $m+1$ is given as follows.

$$\{\hat{E}_i \xrightarrow{\hat{w}_i} \hat{F}_{i-1} \hookrightarrow \hat{F}_i \mid 1 \leq i \leq m+1\} \quad \text{with} \quad \hat{F}_{m+1} = P^m_m \times \Sigma\Omega F'_1, \tag{3.4}$$

where $\hat{E}_{i+1}$, $\hat{F}_i$ and $\hat{w}_{i+1}$, $0 \leq i \leq m$ are defined by

$$\hat{E}_1 = \Omega F_{m-1} \vee \Omega F'_1, \qquad \hat{F}_0 = \{*\}, \qquad \hat{w}_1 = * : \hat{E}_1 \to \hat{F}_0,$$

$$\begin{cases} \hat{E}_{i+1} = E^{i+1}\Omega F_{m-1} \vee \{E^{i}\Omega F_{m-1}{*}\Omega F'_1\}, \\ \hat{F}_i \ = P^{i}\Omega F_{m-1}{\times}\{*\} \cup P^{i-1}\Omega F_{m-1}{\times}\Sigma\Omega F'_1, \\ \hat{w}_{i+1}|_{E^{i+1}\Omega F_{m-1}} : E^{i+1}\Omega F_{m-1} \xrightarrow{p_i^{\Omega F_{m-1}}} P^{i}\Omega F_{m-1}{\times}\{*\} \subset \hat{F}_i, \\ \hat{w}_{i+1}|_{E^{i}\Omega F_{m-1}*\Omega F'_1} = [\chi'_i, 1_{\Sigma\Omega F'_1}]^r : E^{i}\Omega F_{m-1}{*}\Omega F'_1 \to \hat{F}_i, \end{cases} \quad 1{\leq}i{<}m{-}1,$$

$$\begin{cases} \hat{E}_m = \{E^{m}\Omega F_{m-1}{\vee}K_m\} \vee \{E^{m-1}\Omega F_{m-1}{*}\Omega F'_1\}, \\ \hat{F}_{m-1} = P^{m-1}\Omega F_{m-1}{\times}\{*\} \cup P^{m-2}\Omega F_{m-1}{\times}\Sigma\Omega F'_1, \\ \hat{w}_m|_{E^{m}\Omega F_{m-1}\vee K_m} : E^{m}\Omega F_{m-1} \vee K_m \xrightarrow{p'_S} P^{m-1}\Omega F_{m-1}{\times}\{*\} \subset \hat{F}_{m-1}, \\ \hat{w}_m|_{E^{m-1}\Omega F_{m-1}*\Omega F'_1} = [\chi'_{m-1}, 1_{\Sigma\Omega F'_1}]^r : E^{m-1}\Omega F_{m-1}{*}\Omega F'_1 \to \hat{F}_{m-1}, \end{cases} \quad \text{and}$$

$$\begin{cases} \hat{E}_{m+1} = \{E^{m}\Omega F_{m-1}{\vee}K_m\}{*}\Omega F'_1, \\ \hat{F}_m = P^m_m{\times}\{*\} \cup P^{m-1}\Omega F_{m-1}{\times}\Sigma\Omega F'_1, \\ \hat{w}_{m+1} = [\chi'_m, 1_{\Sigma\Omega F'_1}]^r : \hat{E}_{m+1} \to \hat{F}_m, \end{cases}$$

in which $p'_S : E^m\Omega F_{m-1}{\vee}K_m \to P^{m-1}\Omega F_{m-1}$ is given by $p'_S|_{E^m\Omega F_{m-1}} = p_{m-1}^{\Omega F_{m-1}}$ and $p'_S|_{K_m} = q_S$, and χ'_i is a relative homeomorphism given by

$$\begin{cases} \chi'_i : (CE^{i}\Omega F_{m-1}, E^{i}\Omega F_{m-1}) \to (P^{i}\Omega F_{m-1}, P^{i-1}\Omega F_{m-1}), \quad 1{\leq}i{<}m, \\ \chi'_m : (CE', E') \to (P^m_m, P^{m-1}\Omega F_{m-1}),\ E' = E^m\Omega F_{m-1} \vee K_m. \end{cases}$$

From now on, we denote by $\iota^{m,1}_i : F^{m,1}_i \hookrightarrow F^{m,1}_{i+1}$ and $\hat{\iota}_i : \hat{F}_i \hookrightarrow \hat{F}_{i+1}$ the canonical inclusions. Let us denote $1_m = 1_{F_m} : F_m \to F_m$.

Definition 3.2. The identity 1_m is filtered w.r.t. the filtration $* = F_0 \subset F_1 \subset \cdots \subset F_m$. Then by Proposition 2.4 for $f{=}1_m$, we obtain $\sigma_i = \widehat{(1_m)}_i : F_i \to P^i\Omega F_i$ for $1 \leq i \leq m$ and $\widehat{(1_m)}^0_j : K_j \to E^j\Omega F_j$ for $1 \leq j \leq m$. Let $g_j = \widehat{(1_m)}^0_j : K_j \to E^j\Omega F_j$ for $1 \leq j \leq m$. We also obtain $g' = \mathrm{ad}(1_{K'_1}) : K'_1 \to \Omega\Sigma K'_1 = \Omega F'_1$ and $\sigma' = \Sigma g' : F'_1 \to \Sigma\Omega F'_1$.

Since K_m and F_m are of dimension $\ell{-}1$ and ℓ, respectively, we may assume that the images of g_m and σ_m are in $E^m\Omega F_{m-1}$ and P^m_m, respectively.

Lemma 3.3. *Let $\nu^{m,1}_k : F^{m,1}_k \to F^{m,1}_k \vee \Sigma K^{m,1}_k$ and $\hat{\nu}_k : \hat{F}_k \to \hat{F}_k \vee \Sigma\hat{K}_k$ be the canonical co-pairings for $1{\leq}k{\leq}m{+}1$, and $\sigma^{m,1}_m = \sigma_m{\times}\{*\} \cup \sigma_{m-1}{\times}\sigma'$: $F^{m,1}_m \to \hat{F}_m$. Then the following diagram is commutative.*

$$\begin{array}{ccccccc} K^{m,1}_{m+1} & \xrightarrow{w^{m,1}_{m+1}} & F^{m,1}_m & \xrightarrow{\iota^{m,1}_m} & F^{m,1}_{m+1} & \xrightarrow{\nu^{m,1}_{m+1}} & F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1} \\ \downarrow{\scriptstyle g_m*g'} & & \downarrow{\scriptstyle \sigma^{m,1}_m} & & \downarrow{\scriptstyle \sigma_m\times\sigma'} & & \downarrow{\scriptstyle \sigma_m\times\sigma' \vee \Sigma g_m*g'} \\ \hat{E}_{m+1} & \xrightarrow{\hat{w}_{m+1}} & \hat{F}_m & \xrightarrow{\hat{\iota}_m} & \hat{F}_{m+1} & \xrightarrow{\hat{\nu}_{m+1}} & \hat{F}_{m+1} \vee \Sigma\hat{E}_{m+1}. \end{array}$$

As a preparation for showing Lemma 3.3, let us recall the definition of mapping cone $C(h)$ of a given map $h : X \to Z$ and its related spaces.

$$CX = \frac{[0,1]{\times}X}{\{0\}{\times}X},\ C(h) = Z \amalg CX/\sim,\ CX \ni 1{\wedge}x \sim h(x) \in Z,\ x \in X,$$

$$C_{\le\frac{1}{2}}X = \{t{\wedge}x \in CX \,|\, t \le \tfrac{1}{2}\} \approx CX \text{ (natural homeo)},$$

$$C_{\ge\frac{1}{2}}(h) = \{t{\wedge}x \in C(h) \,|\, t \ge \tfrac{1}{2}\},\ \frac{C_{\ge\frac{1}{2}}(h)}{\{\frac{1}{2}\}{\times}X} \approx C(h) \text{ (natural homeo)},$$

where $t{\wedge}x$ denotes the element in CX or $C(h)$, whose representative in $[0,1]{\times}X$ is (t,x). Then we obtain the following propositions.

Proposition 3.4. *Let $K \xrightarrow{a} A \hookrightarrow C(a)$ and $L \xrightarrow{b} B \hookrightarrow C(b)$ be cofibre sequences and let $\nu_a : C(a) \to C(a) \vee \Sigma K$, $\nu_b : C(b) \to C(b) \vee \Sigma L$ and $\nu = \nu(a,b) : C(a){\times}C(b) \to C(a){\times}C(b){\vee}\Sigma K{*}L$ be the canonical co-pairings.*

(1) ν is given by the following composition, natural w.r.t. a and b.

$$C(a){\times}C(b)$$
$$\xrightarrow{\nu_a\times\nu_b} C(a){\times}C(b) \underset{C(a)}{\cup} C(a){\times}\Sigma L \underset{C(b)}{\cup} \Sigma K{\times}C(b) \underset{\Sigma K\vee\Sigma L}{\cup} \Sigma K{\times}\Sigma L$$
$$\xrightarrow{\Phi} C(a){\times}C(b) \vee \Sigma K{\times}\Sigma L/(\Sigma K \vee \Sigma L) \xrightarrow{\approx} C(a){\times}C(b) \vee \Sigma(K{*}L),$$

where Φ is given by $\Phi|_{C(a)\times\Sigma L} = \mathrm{proj}_1$, $\Phi|_{\Sigma K\times C(b)} = \mathrm{proj}_2$ and $\Phi|_{\Sigma K\times\Sigma L} =$ (callpsing) $: \Sigma K \times \Sigma L \twoheadrightarrow \Sigma K{\times}\Sigma L/(\Sigma K{\vee}\Sigma L)$.

(2) Let $K' \xrightarrow{a'} A' \hookrightarrow C(a')$ and $L' \xrightarrow{b'} B' \hookrightarrow C(b')$ be cofibre sequences and $\hat{\nu} = \nu(a',b') : C(a'){\times}C(b') \to C(a'){\times}C(b') \vee \Sigma(K'{*}L')$. If $f^0 : K \to K'$, $f : A \to A'$, $g^0 : L \to L'$ and $g : B \to B'$ satisfy $f{\circ}a = a'{\circ}f^0$ and $g{\circ}b = b'{\circ}g^0$, then (f,f^0) and (g,g^0) induce $f' : C(a) \to C(a')$ and $g' : C(b) \to C(b')$ as in Theorem 2.1, which satisfy $\hat{\nu}{\circ}(f'{\times}g') = (f'{\times}g' \vee \Sigma(f^0{*}g^0)){\circ}\nu : C(a){\times}C(b) \to C(a'){\times}C(b') \vee \Sigma(K'{*}L')$.

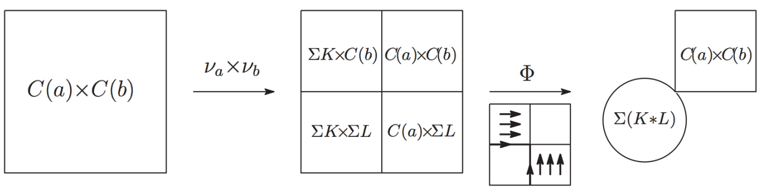


FIGURE 1

Proof. Firstly, we define a homeomorphism

$$\hat{\alpha} : (C(K{*}L), K{*}L) \approx (CK{\times}CL, CK{\times}L \cup K{\times}CL)$$

by $\hat{\alpha}(t{\wedge}(s{\wedge}x,y)) = ((ts){\wedge}x, t{\wedge}y)$ and $\hat{\alpha}(t{\wedge}(x,s{\wedge}y)) = (t{\wedge}x,(ts){\wedge}y)$ for $(x,y) \in K{\times}L$ and $s,t \in [0,1]$ (see Figure 2).

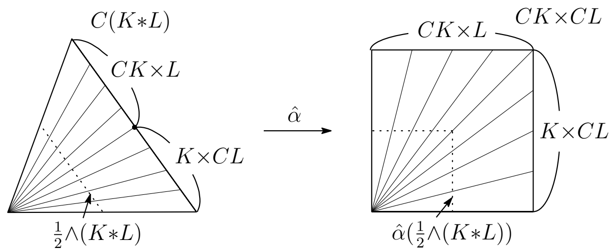


FIGURE 2

Since $C([\chi_a,\chi_b]) = C(a){\times}B \cup A{\times}C(b) \cup_{[\chi_a,\chi_b]} C(K{*}L)$ and $C(a){\times}C(b) = (C(a){\times}B \cup A{\times}C(b)) \cup_{[\chi_a,\chi_b]} CK{\times}CL$, $\hat\alpha$ induces a homeomorphism $\alpha : C([\chi_a,\chi_b]) \approx C(a){\times}C(b)$. Thus the canonical co-pairing ν is given by

$$\nu : C(a){\times}C(b) \to \frac{C(a){\times}C(b)}{\alpha(\{C_{\le\frac12}(K{*}L)\})} \vee \frac{\alpha(C_{\le\frac12}(K{*}L))}{\alpha(\{\frac12\}{\times}(K{*}L))}.$$

Since we can easily see that $\alpha(C_{\le\frac12}(K{*}L))/\alpha(\{\frac12\}{\times}(K{*}L)) \approx \Sigma(K{*}L)$ and $C(a){\times}C(b)/\alpha(\{C_{\le\frac12}(K{*}L)\}) = C(a){\times}C(b)/C_{\le\frac12}K{\times}C_{\le\frac12}L$, ν is given as

$$\nu : C(a){\times}C(b) \to \frac{C(a){\times}C(b)}{C_{\le\frac12}K{\times}C_{\le\frac12}L} \vee \Sigma(K{*}L).$$

Since $C_{\le\frac12}X$ is contractible, the inclusion $(C(a),\{*\}){\times}(C(b),\{*\}) \hookrightarrow (C(a),C_{\le\frac12}K){\times}(C(b),C_{\le\frac12}L)$ is homotopy equivalence, and so is the inclusion $C(a){\times}\{*\} \cup \{*\}{\times}C(b) \hookrightarrow C(a){\times}C_{\le\frac12}L \cup C_{\le\frac12}K{\times}C(b)$.

Hence, the following collapsing map is a homotopy equivalence.

$$\frac{C(a){\times}C_{\le\frac12}L \cup C_{\le\frac12}K{\times}C(b)}{C_{\le\frac12}K{\times}C_{\le\frac12}L} \longrightarrow \frac{C_{\ge\frac12}(a)}{\{\frac12\}{\times}K} \vee \frac{C_{\ge\frac12}(b)}{\{\frac12\}{\times}L} \approx C(a) \vee C(b).$$

Finally, since $C_{\le\frac12}K{\times}C_{\le\frac12}L = \alpha(\{C_{\le\frac12}(K{*}L)\})$, by taking push-out of this collapsing with the inclusion

$$C(a){\times}C_{\le\frac12}L \cup \frac{C_{\le\frac12}K{\times}C(b)}{C_{\le\frac12}K{\times}C_{\le\frac12}L} \hookrightarrow \frac{C(a){\times}C(b)}{\alpha(\{C_{\le\frac12}(K{*}L)\})},$$

we obtain a homotopy equivalence:

$$\frac{C(a){\times}C(b)}{\alpha(\{C_{\le\frac12}(K{*}L)\})} \to \frac{C_{\ge\frac12}(a)}{\{\frac12\}{\times}K} \times \frac{C_{\ge\frac12}(b)}{\{\frac12\}{\times}L} \approx C(a){\times}C(b)$$

Therefore, ν is homotopic to the map $\hat\nu$ which is given by

$$\hat\nu(s{\wedge}x, t{\wedge}y) = \begin{cases} (s{\wedge}x, t{\wedge}y) \in \frac{C_{\ge\frac12}(a)}{\{\frac12\}{\times}K} \times \frac{C_{\ge\frac12}(b)}{\{\frac12\}{\times}L}, & s,t \ge \frac12, \\ (*, t{\wedge}y) \in \{*\} \times \frac{C_{\ge\frac12}(b)}{\{\frac12\}{\times}L}, & s \le \frac12, t \ge \frac12, \\ (s{\wedge}x, *) \in \frac{C_{\ge\frac12}(a)}{\{\frac12\}{\times}K} \times \{*\}, & s \ge \frac12, t \le \frac12, \\ ((s{\wedge}x){\wedge}(t{\wedge}y)) \in \frac{C_{\le\frac12}K}{\{\frac12\}{\times}K} \wedge \frac{C_{\le\frac12}L}{\{\frac12\}{\times}L}, & s,t \le \frac12, \end{cases}$$

which coincides with $\Phi\circ(\nu_a\times\nu_b)$ which implies (1). (2) is clear by concrete definitions of these maps, and we obtain the proposition. ◻

Proposition 3.5. *Let $\nu_m : F_m \to F_m \vee \Sigma K_m$ be the canonical co-pairing and $T_1 : F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m\times F'_1) \vee \Sigma K^{m,1}_{m+1} \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} (\Sigma K_m\times F'_1)$ be an appropriate homeomorphism. Then the following equation holds.*

$$T_1\circ((\nu_m\times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}})\circ\nu^{m,1}_{m+1} = (\nu^{m,1}_{m+1} \cup 1_{\Sigma K_m\times F'_1})\circ(\nu_m\times 1_{F'_1}).$$

Proof. First, Proposition 3.4 implies the following commutative diagram.

$$\begin{array}{ccc}
F_m\times F'_1 & \xrightarrow{\nu^{m,1}_{m+1}} & F_m\times F_1 \vee \Sigma(K_m * K'_1) \\
\big\downarrow{\scriptstyle \nu_m\times 1_{F'_1}} & & \big\uparrow{\scriptstyle \Phi} \\
F_m\times F'_1 \cup_{F'_1} \Sigma K_m\times F'_1 & \xrightarrow{1_m\times\nu_1} & F_m\times F'_1 \cup_{F'_1} \Sigma K_m\times F'_1 \cup_{F_m} F_m\times\Sigma K'_1 \\
 & & \cup\Sigma K_m\times\Sigma K'_1.
\end{array}$$

Since Φ goes through $(F_m\times F'_1 \cup_{F'_1} \Sigma K_m\times F'_1) \cup \Sigma K_m\times\Sigma K'_1/\{*\}\times\Sigma K'_1$ as

$$\begin{aligned}
\Phi : (F_m\times F'_1 \cup_{F'_1} \Sigma K_m\times F'_1 \cup_{F_m} F_m\times\Sigma K'_1) \cup \Sigma K_m\times\Sigma K'_1 \\
\xrightarrow{\Phi'} (F_m\times F'_1 \cup_{F'_1} \Sigma K_m\times F'_1) \cup \frac{\Sigma K_m\times\Sigma K'_1}{\{*\}\times\Sigma K'_1} \\
\xrightarrow{\mathrm{pr}'} F_m\times F'_1 \vee \Sigma(K_m * K'_1),
\end{aligned}$$

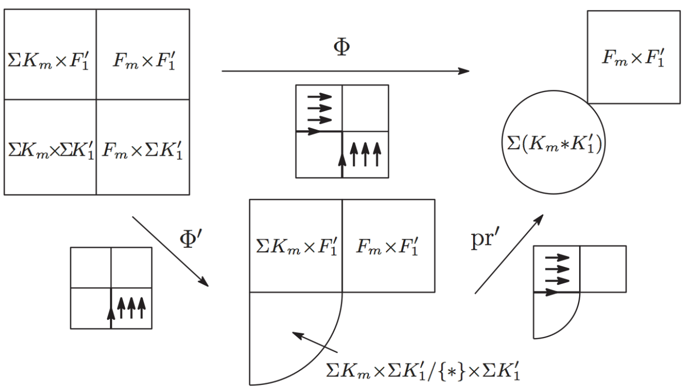


FIGURE 3

where Φ' and pr' are given by the following.

$$\Phi'|_{F_m\times F'_1} = 1_{F_m\times F'_1},\quad \Phi'|_{\Sigma K_m\times F'_1} = 1_{\Sigma K_m\times F'_1},\quad \Phi'|_{F_m\times\Sigma K'_1} = \mathrm{proj}_1,$$

$$\Phi'|_{\Sigma K_m\times\Sigma K'_1} = (\text{collapsing}) : \Sigma K_m\times\Sigma K'_1 \twoheadrightarrow \frac{\Sigma K_m\times\Sigma K'_1}{\{*\}\times\Sigma K'_1}$$

$$\mathrm{pr}'||_{F_m\times F'_1} = 1_{F_m\times F'_1},\quad \mathrm{pr}'|_{\Sigma K_m\times F'_1} = \mathrm{proj}_2,$$

$$\mathrm{pr}'|_{\Sigma K_m\times\Sigma K'_1/\{*\}\times\Sigma K'_1} = (\text{collapsing}) : \frac{\Sigma K_m\times\Sigma K'_1}{\{*\}\times\Sigma K'_1} \twoheadrightarrow \Sigma(K_m * K'_1).$$

Since there is a natural homotopy equivalence $h : \Sigma K_m{\times}\Sigma K'_1/\{*\}{\times}\Sigma K'_1 \simeq \Sigma K_m \vee \Sigma(K_m{*}K'_1)$ such that $h|_{\Sigma K_m\times\{*\}} = 1_{\Sigma K_m}$, pr' can be decomposed as

$$\mathrm{pr}' = \mathrm{pr}'_1{\circ}\mathrm{pr}'_0,$$

where pr'_0 and pr'_1 are given by the following formulae.

$$\mathrm{pr}'_0|_{F_m\times F'_1} = 1_{F_m\times F'_1},\quad \mathrm{pr}'_0|_{\Sigma K_m\times F'_1} = 1_{\Sigma K_m\times F'_1},\quad \mathrm{pr}'_0|_{\Sigma K_m\times\Sigma K'_1/\{*\}\times\Sigma K'_1} = h,$$
$$\mathrm{pr}'_1|_{F_m\times F'_1} = 1_{F_m\times F'_1},\quad \mathrm{pr}'_1|_{\Sigma K_m\times F'_1} = \mathrm{proj}_2,\quad \mathrm{pr}'_1|_{\Sigma(K_m*K'_1)} = 1_{\Sigma(K_m*K'_1)},$$

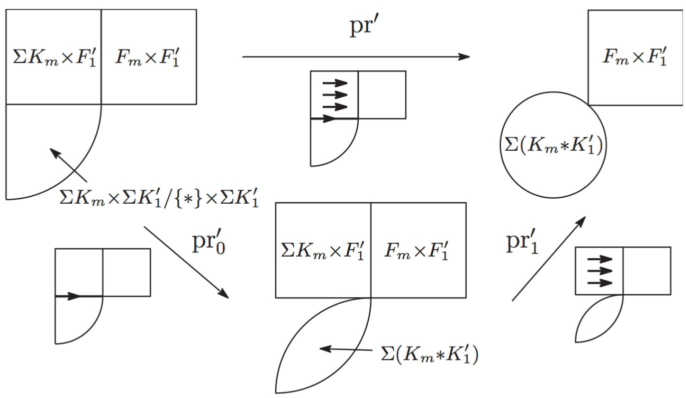


FIGURE 4

Hence Φ is decomposed as $\Phi = \mathrm{pr}'{\circ}\Phi' = \mathrm{pr}'_1{\circ}\mathrm{pr}'_0{\circ}\Phi'$ and $\mathrm{pr}'_0{\circ}\Phi'$ is given by

$$\mathrm{pr}'_0{\circ}\Phi'|_{F_m\times F'_1} = 1_{F_m\times F'_1},\quad \mathrm{pr}'_0{\circ}\Phi'|_{\Sigma K_m\times F'_1} = 1_{\Sigma K_m\times F'_1},$$
$$\mathrm{pr}'_0{\circ}\Phi'|_{F_m\times\Sigma K'_1} = \mathrm{proj}_1 \quad\text{and}$$
$$\mathrm{pr}'_0{\circ}\Phi'|_{\Sigma K_m\times\Sigma K'_1} = (\text{retraction}) : \Sigma K_m{\times}\Sigma K'_1 \to \Sigma K_m \vee \Sigma(K_m{*}K'_1),$$

and hence $\mathrm{pr}'_0{\circ}\Phi'{\circ}(1_m\times\nu_1)$ is given by

$$\mathrm{pr}'_0{\circ}\Phi'{\circ}(1_m\times\nu_1)|_{F_m\times F'_1} = 1_{F_m\times F'_1},$$
$$\mathrm{pr}'_0{\circ}\Phi'{\circ}(1_m\times\nu_1)|_{\Sigma K_m\times F'_1} = \nu' : \Sigma K_m{\times}F'_1 \to \Sigma K_m{\times}F'_1 \vee \Sigma(K_m{*}K'_1),$$

where ν' is the canonical co-pairing. Thus we obtain a commutative diagram

$$\begin{array}{ccc} F^{m,1}_{m+1} = F_m{\times}F'_1 & \xrightarrow{\nu_m\times 1_{F'_1}} & F_m{\times}F'_1 \cup_{F'_1} (\Sigma K_m{\times}F'_1) \\ \downarrow{\scriptstyle \nu^{m,1}_{m+1}} & & \downarrow{\scriptstyle 1_{F_m\times F'_1}\cup\nu'} \\ F_m{\times}F'_1 \vee \Sigma K_m{*}K'_1 & \xleftarrow{p_1} & F_m{\times}F'_1 \cup_{F'_1} (\Sigma K_m{\times}F'_1) \vee \Sigma K_m{*}K'_1. \end{array} \tag{3.5}$$

Therefore we have

$$T_1{\circ}((\nu_m\times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}){\circ}\nu^{m,1}_{m+1}$$
$$= T_1{\circ}((\nu_m\times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}){\circ}p_1{\circ}(1_{F^{m,1}_{m+1}} \cup\nu'){\circ}(\nu_m\times 1_{F'_1}).$$

Let us denote by $p_2 : F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m{\times}F'_1) \cup_{F'_1} (\Sigma K_m{\times}F'_1) \vee \Sigma K^{m,1}_{m+1} \to F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m{\times}F'_1) \vee \Sigma K^{m,1}_{m+1}$ the map pinching the second $\Sigma K_m{\times}F'_1$ to

F'_1, by $p_3 : F^{m,1}_{m+1} \cup_{F'_1} ((\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} (\Sigma K_m \times F'_1) \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} \Sigma K^{m,1}_{m+1}$ the map pinching the first $\Sigma K_m \times F'_1$ to one point, by $\nu_0 : \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ the canonical co-multiplication and by $T_0 : \Sigma K_m \vee \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ the switching map. It is then easy to check

$$\begin{aligned}
&T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\
&\quad = T_1 \circ p_2 \circ ((\nu_m \times 1_{F'_1}) \cup 1_{\Sigma K_m \times F'_1} \vee 1_{\Sigma K_m * K'_1}) \circ (1_{F^{m,1}_{m+1}} \cup \nu') \circ (\nu_m \times 1_{F'_1}) \\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ (1_{F^{m,1}_{m+1}} \cup (T_0 \times 1_{F'_1})) \\
&\qquad\qquad \circ ((\nu_m \times 1_{F'_1}) \cup 1_{\Sigma K_m \times F'_1}) \circ (\nu_m \times 1_{F'_1}).
\end{aligned}$$

Using $(1_{F_m} \vee \nu_0) \circ \nu_m = (\nu_m \vee 1_{\Sigma K_m}) \circ \nu_m$ and $T_0 \circ \nu_0 = \nu_0$ from the assumption that K_m is a co-H-space together with $F^{m,1}_{m+1} = F_m \times F'_1$, we have

$$\begin{aligned}
&T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ (1_{F^{m,1}_{m+1}} \cup (T_0 \times 1_{F'_1})) \\
&\qquad\qquad \circ (1_{F^{m,1}_{m+1}} \cup (\nu_0 \times 1_{F'_1})) \circ (\nu_m \times 1_{F'_1}) \\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ ((1_{F_m} \vee \nu_0) \times 1_{F'_1})) \circ (\nu_m \times 1_{F'_1}) \\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ ((\nu_m \vee 1_{\Sigma K_m}) \times 1_{F'_1}) \circ (\nu_m \times 1_{F'_1}).
\end{aligned}$$

Using the diagram (3.5), we proceed further as follows:

$$T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} = (\nu^{m,1}_{m+1} \cup 1_{\Sigma K_m \times F'_1}) \circ (\nu_m \times 1_{F'_1}).$$

It completes the proof of Proposition 3.5. □

Proof of Lemma 3.3. The commutativity of the left square follows from [14, Proposition 2.9] and the middle square is clearly commutative.

So we are left to show $(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1} = \hat{\nu}_{m+1} \circ (\sigma_m \times \sigma')$. Recall that $\sigma_m = \widehat{1_m}$ which is given by Proposition 2.4 (1) for $f = 1_m$. On the other hand by Proposition 2.4 (2), we have $\sigma_m = \nabla_{P^m \Omega F_m} \circ ((1_m)'_m \vee \iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \circ \nu_m$, and hence we obtain

$$\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1} \\
&\quad = \{(\nabla_{P^m \Omega F_m} \circ ((1_m)'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m)) \circ \nu_m) \times \sigma' \vee \Sigma g_m * g'\} \circ \nu^{m,1}_{m+1} \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma \Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}}) \\
&\qquad\quad \circ \{((1_m)'_m \times \sigma') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m) \times \sigma') \vee \Sigma g_m * g'\} \\
&\qquad\quad \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma \Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}}) \\
&\qquad\quad \circ T_2 \circ \{((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \circ T_1 \\
&\qquad\quad \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1},
\end{aligned}$$

where $T_1 : F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1} \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} (\Sigma K_m \times F'_1)$ and $T_2 : (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \cup_{\Sigma\Omega F'_1} \hat{F}_{m+1} \to (\hat{F}_{m+1} \cup_{\Sigma\Omega F'_1} \hat{F}_{m+1}) \vee \Sigma \hat{E}_{m+1}$ are appropriate homeomorphisms. Then by Proposition 3.5, Proposition 3.4 (2) and the definitions of $(1_m)'_m$ and σ', we proceed as follows.

$$
\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1} \\
&\quad = (\nabla_{P^m\Omega F_m} \times 1_{\Sigma\Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}}) \\
&\qquad\quad \circ T_2 \circ \{((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \\
&\qquad\quad \circ (\nu^{m,1}_{m+1} \cup 1_{\Sigma K_m \times F'_1}) \circ (\nu_m \times 1_{F'_1}) \\
&\quad = (\nabla_{P^m\Omega F_m} \times 1_{\Sigma\Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}}) \circ T_2 \\
&\qquad\quad \circ \{(((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1}) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \circ (\nu_m \times 1_{F'_1}). \\
&\quad = (\nabla_{P^m\Omega F_m} \times 1_{\Sigma\Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}}) \circ T_2 \\
&\qquad\quad \circ \{(\hat{\nu}_{m+1} \circ ((1_m)'_m \times \sigma')) \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \circ (\nu_m \times 1_{F'_1}) \\
&\quad = (\nabla_{P^m\Omega F_m} \times 1_{\Sigma\Omega F'_1} \vee \nabla_{\Sigma \hat{E}_{m+1}}) \circ T_3 \\
&\qquad\quad \circ \{\hat{\nu}_{m+1} \circ ((1_m)'_m \times \sigma') \cup i_1 \circ ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \circ (\nu_m \times 1_{F'_1}).
\end{aligned}
$$

Here $i_1 : \hat{F}_{m+1} \to \hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}$ is the first inclusion and $T_3 : (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \cup_{\Sigma\Omega F'_1} (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \to (\hat{F}_{m+1} \cup_{\Sigma\Omega F'_1} \hat{F}_{m+1}) \vee \Sigma \hat{E}_{m+1} \vee \Sigma \hat{E}_{m+1}$ is the appropriate homeomorphism. Thus we proceed further as follows.

$$
\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1} \\
&\quad = (\nabla_{P^m\Omega F_m} \times 1_{\Sigma\Omega F'_1} \vee \nabla_{\Sigma \hat{E}_{m+1}}) \circ T_3 \\
&\qquad\quad \circ (\hat{\nu}_{m+1} \cup \hat{\nu}_{m+1}) \circ \{((1_m)'_m \times \sigma') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m) \times \sigma')\} \circ (\nu_m \times 1_{F'_1}) \\
&\quad = \hat{\nu}_{m+1} \circ \{\nabla_{P^m\Omega F_m} \circ ((1_m)'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m)) \circ \nu_m \times \sigma'\} = \hat{\nu}_{m+1} \circ (\sigma^{1_m}_m \times \sigma').
\end{aligned}
$$

It completes the proof of Lemma 3.3. □

4. Proof of Theorem 1.2

In the fibre sequence $G \hookrightarrow E \to \Sigma A$, by the James-Whitehead decomposition (see Whitehead [17, VII. Theorem (1.15)]), the total space E has the homotopy type of the space $G \cup_\psi G \times CA$. Here ψ is the following map.

$$\psi : G \times A \xrightarrow{1_G \times \alpha} G \times G \xrightarrow{\mu} G.$$

Since $G \simeq F_m$ and, by the condition (1) of Theorem 1.2, α is compressible into F'_1. Hence we see that

$$\psi : G \times A \simeq F_m \times A \xrightarrow{1_{F_m} \times \alpha} F_m \times F'_1 \subset F_m \times F_1 \subset F_m \times F_m \simeq G \times G \xrightarrow{\mu} G \simeq F_m$$

and E is the homotopy pushout of the following sequence.

$$F_m \xleftarrow{pr_1} F_m \times A \xrightarrow{1_{F_m} \times \alpha} F_m \times F'_1 \xrightarrow{\mu_{m,1}} F_m.$$

We construct spaces and maps such that the homotopy pushout of these maps dominates E. Let $e' = e_1^{F_1'} : \Omega\Sigma F_1' \to F_1'$ and $\sigma_A = \Sigma\,\mathrm{ad}(1_A) : A \to \Sigma\Omega A$, since A is a suspended space. By the condition (2) of Theorem 1.2, we have $H_1(\alpha) = 0$ in $[A, \Omega F_1' * \Omega F_1']$, which immediately implies

$$\sigma'\circ\alpha = \Sigma\,\mathrm{ad}(\alpha) = e'\circ\sigma_A : A \to \Sigma\Omega F_1'. \tag{4.1}$$

By the condition (3) of Theorem 1.2, we have $K_m = S^{\ell-1}$ with $m \geq 3$ and $\ell \geq 3$, and so $(P^m\Omega F_m, P^m_m)$ is ℓ-connected by Lemma 3.1.

Proposition 4.1. *The following diagram is commutative.*

$$\begin{array}{ccccccc}
F_m & \xleftarrow{pr_1} & F_m\times A & \xrightarrow{1_{F_m}\times\alpha} & F_m\times F_1' & \xrightarrow{\mu_{m,1}} & F_m \\
\downarrow{\scriptstyle \iota^{\Omega F_m}_{m,m+1}\circ\sigma_m} & & \downarrow{\scriptstyle \sigma_m\times\sigma_A} & & \downarrow{\scriptstyle \sigma_m\times\sigma'} & & \downarrow{\scriptstyle \iota^{\Omega F_m}_{m,m+1}\circ\sigma_m} \\
P^{m+1}\Omega F_m & \xleftarrow{\phi} & P^m_m\times\Sigma\Omega A & \xrightarrow{\chi} & \hat{F}_{m+1} & & P^{m+1}\Omega F_m \\
\downarrow{\scriptstyle e^{F_m}_{m+1}} & & \downarrow{\scriptstyle e^{F_m}_m\times e^A_1} & & \downarrow{\scriptstyle e^{F_m}_m\times e'} & & \downarrow{\scriptstyle e^{F_m}_{m+1}} \\
F_m & \xleftarrow[pr_1]{} & F_m\times A & \xrightarrow[1_{F_m}\times\alpha]{} & F_m\times F_1' & \xrightarrow[\mu_{m,1}]{} & F_m,
\end{array}$$

where $\phi = \iota^{\Omega F_m}_{m,m+1}\circ pr_1$ *and* $\chi = 1_{P^m_m}\times\Sigma\Omega\alpha$.

Proof. The left upper square is clearly commutative. The equation $e^{F_m}_m = e^{F_m}_{m+1}\circ\iota^{\Omega F_m}_{m,m+1}$ implies that the left lower square is commutative. The equation $\alpha\circ e^A_1 = e'\circ\Sigma\Omega\alpha$ implies the commutativity of the middle lower square. The commutativity of the middle upper square is obtained by (4.1). By Proposition 2.4 (2) for $f = 1_m$ and the fact $e'\circ\sigma' = 1_{F_1'}$ imply that the right rectangular is commutative. It completes the proof of the proposition. $\square$

Definition 4.2. $\lambda = \mu_{m,1}\circ\{e^{F_m}_m\times e'\} : \hat{F}_{m+1} \to F_m\times F_1' \to F_m$.

Then λ is a well-defined filtered map w.r.t. the filtration (3.4) of $\hat{F}_{m+1}$ and the trivial filtration ($(F_m)_i = F_m$ for all i) of F_m, where $\{e^{F_m}_m\times e'\}(\hat{F}_k) = \{e^{F_{m-1}}_k\times * \cup e^{F_{m-1}}_{k-1}\times e'\}(\hat{F}_k) \subset F_{m-1}\times F_1'$ for $0 \leq k < m$, and $\{e^{F_m}_m\times e'\}(\hat{F}_m) = \{e^{F_m}_m\times * \cup e^{F_{m-1}}_{m-1}\times e'\}(\hat{F}_m) \subset F_m\times\{*\} \cup F_{m-1}\times F_1'$ for $k = m$.

Definition 4.3. By Proposition 2.4 for $f = \lambda$, we obtain a series of maps $\hat{\lambda}_k : \hat{F}_k \to P^k\Omega F_m,\ 0 \leq k \leq m+1$.

By the hypothesis of Theorem 1.2, we have $\mu_{k,1} : F_k\times F_1' \to F_{k+1}$ for $k < m$, and $\mu_{m,1} : F_m\times F_1' \to F_m$, both of which are restrictions of μ.

Lemma 4.4. *There is a map* $\hat{\lambda} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$ *which fits in with the following commutative diagram obtained by dividing the right square of the* *diagram in Proposition 4.1 by $\hat{\lambda}$ into upper and lower squares.*

$$\begin{array}{ccccccc}
F_m & \xleftarrow{pr_1} & F_m{\times}A & \xrightarrow{1\times\alpha} & F_m{\times}F'_1 & \xrightarrow{\mu_{m,1}} & F_m \\
\downarrow{\scriptstyle \iota^{\Omega F_m}_{m,m+1}\circ\sigma_m} & & \downarrow{\scriptstyle \sigma_m\times\sigma_A} & & \downarrow{\scriptstyle \sigma_m\times\sigma'} & & \downarrow{\scriptstyle \iota^{\Omega F_m}_{m,m+1}\circ\sigma_m} \\
P^{m+1}\Omega F_m & \xleftarrow{\phi} & P^m_m{\times}\Sigma\Omega A & \xrightarrow{\chi} & \hat{F}_{m+1} & \xrightarrow{\hat{\lambda}} & P^{m+1}\Omega F_m \\
\downarrow{\scriptstyle e^{F_m}_{m+1}} & & \downarrow{\scriptstyle e^{F_m}_m\times e^A_1} & & \downarrow{\scriptstyle e^{F_m}_m\times e'} & & \downarrow{\scriptstyle e^{F_m}_{m+1}} \\
F_m & \xleftarrow[pr_1]{} & F_m{\times}A & \xrightarrow[1\times\alpha]{} & F_m{\times}F'_1 & \xrightarrow[\mu_{m,1}]{} & F_m.
\end{array}$$

Proof. Let $\mu^{m,1}_k = 1_{F_k}\cup\mu_{k-1,1} : F^{m,1}_k = F_k{\times}\{*\}\cup F_{k-1}{\times}F'_1 \to F_k$, $\sigma^{m,1}_k = \sigma_k{\times}{*} \cup \sigma_{k-1}{\times}\sigma' : F^{m,1'}_k \to P^k\Omega F_k{\times}\{*\}\cup P^{k-1}\Omega F_{k-1}{\times}\Sigma\Omega F'_1$ and $j_k = P^k\Omega i^F_{k,m-1}{\times}*\cup P^{k-1}\Omega i^F_{k-1,m-1}{\times}1_{\Sigma\Omega F'_1}$, $0{\leq}k{<}m$.

Firstly, we show the following equation by induction on $k{<}m$.

$$\iota^{\Omega F_m}_{k,k+1}{\circ}P^k\Omega i^F_{k,m}{\circ}\sigma_k{\circ}\mu^{m,1}_k = \iota^{\Omega F_m}_{k,k+1}{\circ}\hat{\lambda}_k{\circ}j_k{\circ}\sigma^{m,1}_k : F^{m,1}_k \to P^{k+1}\Omega F_m. \tag{4.2}$$

The case $k=0$ is clear, since both maps are constant maps. Assume the k-th of (4.2). By Proposition 2.4 (1) for $f{=}1_m$, the diagram

$$\begin{array}{ccccccc}
F_k & \xrightarrow{\sigma_k} & P^k\Omega F_k & \xhookrightarrow{P^k\Omega i^F_{k,k+1}} & P^k\Omega F_{k+1} & \xhookrightarrow{P^k\Omega i^F_{k+1,m-1}} & P^k\Omega F_{m-1} \\
\downarrow{\scriptstyle i^F_{k,k+1}} & & & & \downarrow{\scriptstyle \iota^{\Omega F_{k+1}}_{k,k+1}} & & \downarrow{\scriptstyle \iota^{\Omega F_{m-1}}_{k,k+1}} \\
F_{k+1} & & \xrightarrow[\sigma_{k+1}]{} & & P^{k+1}\Omega F_{k+1} & \xhookrightarrow[P^{k+1}\Omega i^F_{k+1,m-1}]{} & P^{k+1}\Omega F_{m-1}
\end{array}$$

is commutative for $k{+}1{<}m$, and hence we have

$$\begin{aligned}
&j_{k+1}{\circ}\sigma^{m,1}_{k+1}{\circ}\iota^{m,1}_k \\
&\quad= (P^{k+1}\Omega i^F_{k+1,m-1}{\circ}\sigma_{k+1}{\circ}i^F_{k,k+1}){\times}*\cup(P^k\Omega i^F_{k,m-1}{\circ}\sigma_k{\circ}i^F_{k-1,k}){\times}\sigma' \\
&\quad= (\iota^{\Omega F_{m-1}}_{k,k+1}{\circ}P^k\Omega i^F_{k,m-1}{\circ}\sigma_k){\times}*\cup(\iota^{\Omega F_{m-1}}_{k-1,k}{\circ}P^k\Omega i^F_{k-1,m-1}{\circ}\sigma_{k-1}){\times}\sigma' \\
&\quad= \hat{\iota}_k{\circ}j_k{\circ}\sigma^{m,1}_k.
\end{aligned}$$

By Proposition 2.4 (1) for $f{=}\lambda$, we have $\hat{\lambda}_{k+1}{\circ}\hat{\iota}_k = \iota^{\Omega F_m}_{k,k+1}{\circ}\hat{\lambda}_k$, and hence

$$\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1}{\circ}\iota^{m,1}_k = \hat{\lambda}_{k+1}{\circ}\hat{\iota}_k{\circ}j_k{\circ}\sigma^{m,1}_k = \iota^{\Omega F_m}_{k,k+1}{\circ}\hat{\lambda}_k{\circ}j_k{\circ}\sigma^{m,1}_k.$$

Then, by Proposition 2.4 (1) for $f = 1_m$ and the induction hypothesis, we proceed further as follows.

$$\begin{aligned}
&(\iota^{m,1}_k)^*(\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1}) \\
&\quad= [\iota^{\Omega F_m}_{k,k+1}{\circ}P^k\Omega i^F_{k,m}{\circ}\sigma_k{\circ}\mu^{m,1}_k] = [P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}i^F_{k,k+1}{\circ}\mu^{m,1}_k] \\
&\quad= [P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1}{\circ}\iota^{m,1}_k] = (\iota^{m,1}_k)^*(P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1}).
\end{aligned}$$

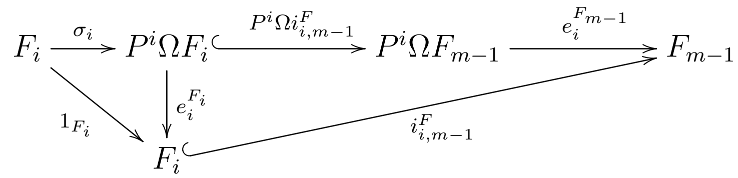
By a standard argument of homotopy theory on a cofibre sequence $K^{m,1}_{k+1} \to F^{m,1}_{k} \hookrightarrow F^{m,1}_{k+1}$, we obtain the difference map $\delta_{k+1} : \Sigma K^{m,1}_{k+1} \to P^{k+1}\Omega F_m$ of $\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1}$ and $P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1}$, $k{+}1{<}m$:

$$(4.3)\quad P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1} = \nabla_{P^{k+1}\Omega F_m}{\circ}(\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} \vee \delta_{k+1}){\circ}\nu^{m,1}_{k+1}.$$

Then, by Proposition 2.4 (1) for $f = \lambda$, we have

$$e^{F_m}_{k+1}{\circ}\hat{\lambda}_{k+1} = \mu_{m-1,1}{\circ}\{e^{F_{m-1}}_{k+1}{\times}{*} \cup e^{F_{m-1}}_{k}{\times}e'\},$$

and hence, by the commutative diagram



for $i = k,\ k{+}1 \leq m{-}1$, we obtain the equation

$$\{e^{F_{m-1}}_{k+1}{\times}{*} \cup e^{F_{m-1}}_{k}{\times}e'\}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} = \iota^{m,1}_{k+1,m},$$

where $\iota^{m,1}_{k+1,m} : F^{m,1}_{k+1} \hookrightarrow F^{m,1}_{m}$ is the canonical inclusion. Thus we have

$$\begin{aligned} e^{F_m}_{k+1}{\circ}\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} &= \mu_{m-1,1}{\circ}\iota^{m,1}_{k+1,m} = i^F_{k+1,m}{\circ}\mu^{m,1}_{k+1} \\ &= i^F_{k+1,m}{\circ}e^{F_{k+1}}_{k+1}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1} = e^{F_m}_{k+1}{\circ}P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1}, \end{aligned}$$

and hence, by (4.3), we obtain

$$\begin{aligned} i^F_{k+1,m}{\circ}\mu^{m,1}_{k+1} &= \nabla_{F_m}{\circ}(e^{F_m}_{k+1}{\circ}\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} \vee e^{F_m}_{k+1}{\circ}\delta_{k+1}){\circ}\nu^{m,1}_{k+1} \\ &= \nabla_{F_m}{\circ}(i^F_{k+1,m}{\circ}\mu^{m,1}_{k+1} \vee e^{F_m}_{k+1}{\circ}\delta_{k+1}){\circ}\nu^{m,1}_{k+1}. \end{aligned}$$

Using [13, Theorem 2.7 (1)] and the multiplication μ on $G \simeq F_m$, $e^{F_m}_{k+1}{\circ}\delta_{k+1} : \Sigma K^{m,1}_{k+1} \to F_m$ is null-homotopic. Hence by a standard argument of homotopy theory on the fibre sequence $E^{k+2}\Omega F_m \to P^{k+1}\Omega F_m \to F_m$, we obtain a lift $\delta'_{k+1} : \Sigma K^{m,1}_{k+1} \to E^{m+1}\Omega F_m$ of δ_{k+1} as $p^{\Omega F_m}_{k+1}{\circ}\delta'_{k+1} = \delta_{k+1}$, $k{+}1 < m$. Since $\iota^{\Omega F_m}_{k+1,k+2}{\circ}p^{\Omega F_m}_{k+1} = *$, we obtain $\iota^{\Omega F_m}_{k+1,k+2}{\circ}\delta_{k+1} = \iota^{\Omega F_m}_{k+1,k+2}{\circ}p^{\Omega F_m}_{k+1}{\circ}\delta'_{k+1} = *$ and

$$\begin{aligned} &\iota^{\Omega F_m}_{k+1,k+2}{\circ}\nabla_{P^{k+1}\Omega F_m}{\circ}(\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} \vee \delta_{k+1}){\circ}\nu^{m,1}_{k+1} \\ &\quad= \nabla_{P^{k+2}\Omega F_m}{\circ}(\iota^{\Omega F_m}_{k+1,k+2}{\circ}\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1} \vee *){\circ}\nu^{m,1}_{k+1} \\ &\quad= \iota^{\Omega F_m}_{k+1,k+2}{\circ}\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1}, \end{aligned}$$

and hence, by (4.3), we obtain

$$\iota^{\Omega F_m}_{k+1,k+2}{\circ}P^{k+1}\Omega i^F_{k+1,m}{\circ}\sigma_{k+1}{\circ}\mu^{m,1}_{k+1} = \iota^{\Omega F_m}_{k+1,k+2}{\circ}\hat{\lambda}_{k+1}{\circ}j_{k+1}{\circ}\sigma^{m,1}_{k+1}.$$

It completes the proof of the induction step and we obtain (4.2) for $k{<}m$.

Secondly, we show the following equation

$$(4.4)\qquad \iota^{\Omega_{F_m}}_{m,m+1}{\circ}\sigma_m{\circ}\mu^{m,1}_m = \iota^{\Omega_{F_m}}_{m,m+1}{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m.$$

By Proposition 2.4 (1) for $f = 1_m$, we obtain

$$\sigma_t{\circ}i^F_{t-1,t} = i^{\Omega F_t}_{t-1,t}{\circ}P^{t-1}\Omega i^F_{t-1,t}{\circ}\sigma_{t-1} \qquad \text{for} \quad t = m{-}1, m.$$

Hence we have

$$\begin{aligned}\sigma^{m,1}_m{\circ}\iota^{m,1}_{m-1} &= ((\sigma_m{\circ}i^F_{m-1,m}){\times}*\cup(\sigma_{m-1}{\circ}i^F_{m-2,m-1}){\times}\sigma')\\ &= (\iota^{\Omega F_m}_{m-1,m}{\circ}P^{m-1}\Omega i^F_{m-1,m}{\circ}\sigma_{m-1}){\times}*\\ &\qquad\qquad \cup\ (\iota^{\Omega F_{m-1}}_{m-2,m-1}{\circ}P^{m-2}\Omega i^F_{m-2,m-1}{\circ}\sigma_{m-1}){\times}\sigma'\\ &= \hat{i}_{m-1}{\circ}j_{m-1}{\circ}\sigma^{m,1}_{m-1}.\end{aligned}$$

By Proposition 2.4 (1) for $f = \lambda$, we obtain $\hat{\lambda}_m{\circ}\hat{\iota}_{m-1} = \iota^{\Omega F_m}_{m-1,m}{\circ}\hat{\lambda}_{m-1}$ and

$$\begin{aligned}(\iota^{m,1}_{m-1})^*(\hat{\lambda}_m{\circ}\sigma^{m,1}_m) &= [\hat{\lambda}_m{\circ}\sigma^{m,1}_m{\circ}\iota^{m,1}_{m-1}] = [\hat{\lambda}_m{\circ}\hat{i}_{m-1}{\circ}j_{m-1}{\circ}\sigma^{m,1}_{m-1}]\\ &= [\iota^{\Omega F_m}_{m-1,m}{\circ}\hat{\lambda}_{m-1}{\circ}j_{m-1}{\circ}\sigma^{m,1}_{m-1}] = [\iota^{\Omega F_m}_{m-1,m}{\circ}P^m\Omega i^F_{m-1,m}{\circ}\sigma_{m-1}{\circ}\mu^{m,1}_{m-1}]\\ &= [\sigma_m{\circ}i^F_{m-1,m}{\circ}\mu^{m,1}_{m-1}] = (\iota^{m,1}_{m-1})^*(\sigma_m{\circ}\mu^{m,1}_m)\end{aligned}$$

using (4.2) for $k = m{-}1$. Thus by a standard argument of homotopy theory on the cofibre sequence $K^{m,1}_m \to F_m \hookrightarrow F_{m+1}$, we obtain a difference map $\delta_m : \Sigma K^{m,1}_m \to P^m\Omega F_m$ of $\hat{\lambda}_m{\circ}\sigma^{m,1}_m$ and $\sigma_m{\circ}\mu^{m,1}_m$:

$$\sigma_m{\circ}\mu^{m,1}_m = \nabla_{P^m\Omega F_m}{\circ}(\hat{\lambda}_m{\circ}\sigma^{m,1}_m \vee \delta_m){\circ}\nu^{m,1}_m. \tag{4.5}$$

By Proposition 2.4 (1) for $f{=}\lambda$, we have the equation

$$e^{F_m}_m{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m = \mu^{m,1}_m{\circ}\{e^{F_m}_m{\times}*\cup e^{F_{m-1}}_{m-1}{\times}e'\}{\circ}(\sigma_m{\times}*\cup\sigma_{m-1}{\times}\sigma') = \mu^{m,1}_m,$$

and hence, by (4.5), we obtain

$$\mu^{m,1}_m = \nabla_{F_m}{\circ}(e^{F_m}_m{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m \vee e^{F_m}_m{\circ}\delta_m){\circ}\nu^{m,1}_m = \nabla_{F_m}{\circ}(\mu^{m,1}_m \vee e^{F_m}_m{\circ}\delta_m){\circ}\nu^{m,1}_m.$$

Thus we obtain $e^{F_m}_m{\circ}\delta_m = *$. Then, by a standard argument in homotopy theory on the fibre sequence $E^{m+1}\Omega F_m \to P^m\Omega F_m \to F_m$, we obtain a lift $\delta'_m : \Sigma K^{m,1}_m \to E^{m+1}\Omega F_m$ which satisfies $\delta_m = p^{\Omega F_m}_m{\circ}\delta'_m$. Since $\iota^{\Omega F_m}_{m,m+1}{\circ}p^{\Omega F_m}_m = *$, we have $\iota^{\Omega F_m}_{m,m+1}{\circ}\delta_m = \iota^{\Omega F_m}_{m,m+1}{\circ}p^{\Omega F_m}_m{\circ}\delta'_m = *$. Then by (4.5), we obtain (4.4) as follows:

$$\begin{aligned}\iota^{\Omega F_m}_{m,m+1}{\circ}\sigma_m{\circ}\mu^{m,1}_m &= \iota^{\Omega F_m}_{m,m+1}{\circ}\nabla_{P^m\Omega F_m}{\circ}(\hat{\lambda}_m{\circ}\sigma^{m,1}_m \vee \delta_m){\circ}\nu^{m,1}_m\\ &= \nabla_{P^{m+1}\Omega F_m}{\circ}(\iota^{\Omega F_m}_{m,m+1}{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m \vee *){\circ}\nu^{m,1}_m = \iota^{\Omega F_m}_{m,m+1}{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m.\end{aligned}$$

Finally, we construct a map $\hat{\lambda} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$. By Proposition 2.4 (1) for $f{=}1_m$, we have $\sigma_m{\circ}i^F_{m-1,m} = i^{\Omega F_m}_{m-1,m}{\circ}P^{m-1}\Omega i^F_{m-1,m}{\circ}\sigma_{m-1}$, and hence

$$\begin{aligned}(\sigma_m{\times}\sigma'){\circ}\iota^{m,1}_m &= (\sigma_m{\times}\sigma'){\circ}(1_{F_m}\times*\cup i^F_{m-1,m}{\times}1_{F'_1})\\ &= \hat{\iota}_m{\circ}(\sigma_m{\times}*\cup\sigma_{m-1}{\times}\sigma') = \hat{\iota}_m{\circ}\sigma^{m,1}_m.\end{aligned}$$

Also by Proposition 2.4 (1) for $f{=}\lambda$, we obtain $\hat{\lambda}_{m+1}{\circ}\hat{\iota}_m = \iota^{\Omega F_m}_{m,m+1}{\circ}\hat{\lambda}_m$ and

$$\hat{\lambda}_{m+1}{\circ}(\sigma_m{\times}\sigma'){\circ}\iota^{m,1}_m = \hat{\lambda}_{m+1}{\circ}\hat{\iota}_m{\circ}\sigma^{m,1}_m = \iota^{\Omega F_m}_{m,m+1}{\circ}\hat{\lambda}_m{\circ}\sigma^{m,1}_m,$$

and hence, by (4.4), we obtain

$$(\iota^{m,1}_m)^*(\hat\lambda_{m+1}\circ(\sigma_m\times\sigma')) = \iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu^{m,1}_m = (\iota^{m,1}_m)^*(\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu_{m,1}).$$

By a standard argument of homotopy theory on a cofibre sequence $K^{m,1}_{m+1} \to F^{m,1}_m \hookrightarrow F^{m,1}_{m+1}$, we obtain a map $\delta_{m+1} : \Sigma K^{m,1}_{m+1} \to P^{m+1}\Omega F_m$ such that

$$\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu_{m,1} = \nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1}\circ(\sigma_m\times\sigma') \vee \delta_{m+1})\circ\nu^{m,1}_{m+1}. \tag{4.6}$$

To proceed further, let us consider the dotted map $\bar e : \Sigma\hat E_{m+1} \to \Sigma K^{m+1}_m$, induced from the commutativity of the lower left square, in the diagram

$$\begin{array}{ccccc}
F^{m,1}_m & \xhookrightarrow{\iota^{m,1}_m} & F^{m,1}_{m+1} & \xrightarrow{q_P} & \Sigma K^{m,1}_m \\
\downarrow{\scriptstyle \sigma^{m,1}_m} & & \downarrow{\scriptstyle \sigma_m\times\sigma'} & & \downarrow{\scriptstyle \Sigma g_m * g'} \\
\hat F_m & \xhookrightarrow{\hat\iota_m} & \hat F_{m+1} & \xrightarrow{\bar q_F} & \Sigma\hat E_{m+1} \\
\downarrow{\scriptstyle \hat e_m} & & \downarrow{\scriptstyle e^{F_m}_m\times e'} & & \downarrow{\scriptstyle \bar e} \\
F^{m,1}_m & \xhookrightarrow{\iota^{m,1}_m} & F^{m,1}_{m+1} & \xrightarrow{q_P} & \Sigma K^{m,1}_m,
\end{array}$$

where the map $\hat e_m : \hat F_m \to F^{m,1}_m$ is $e^{F_m}_m\times * \cup e^{F_{m-1}}_{m-1}\times e'$. Since $\hat e_m\circ\sigma^{m,1}_m$ and $(e^{F_m}_m\times e')\circ(\sigma_m\times\sigma')$ are homotopy equivalences, $\bar e\circ\Sigma g_m*g_1$ is also a homotopy equivalence (see [4, Lemma 16.24]). We denote by $h : \Sigma K^{m+1}_m \to \Sigma K^{m+1}_m$ the homotopy inverse of $\bar e\circ\Sigma g_m*g_1$. Then, by (4.6), we obtain

$$\begin{aligned}
\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu_{m,1} &= \nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1}\circ(\sigma_m\times\sigma') \vee \delta_{m+1})\circ\nu^{m,1}_{m+1} \\
&= \nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1}\circ(\sigma_m\times\sigma') \vee \delta_{m+1}\circ h\circ\bar e\circ\Sigma g_m*g')\circ\nu^{m,1}_{m+1} \\
&= \nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1} \vee \delta_{m+1}\circ h\circ\bar e)\circ((\sigma_m\times\sigma') \vee \Sigma g_m*g')\circ\nu^{m,1}_{m+1},
\end{aligned}$$

and hence, by Lemma 3.3, we proceed further as

$$= \nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1} \vee \delta_{m+1}\circ h\circ\bar e)\circ\hat\nu_{m+1}\circ(\sigma_m\times\sigma').$$

This suggest us to define $\hat\lambda$ by $\nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1}\vee\delta_{m+1}\circ h\circ\bar e)\circ\hat\nu_{m+1}$ to obtain

$$\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu_{m,1} = \hat\lambda\circ(\sigma_m\times\sigma') : F_m\times F'_1 \to P^{m+1}\Omega F_m,$$

which gives the commutativity of the upper right square in Lemma 4.4. So we are left to show the commutativity of the lower right square in Lemma 4.4: by Proposition 2.4 (1) for $f = \lambda$, we have

$$e^{F_m}_{m+1}\circ\hat\lambda_{m+1}\circ(\sigma_m\times\sigma') = \mu_{m,1}\circ\{e^{F_m}_m\times e'\}\circ(\sigma_m\times\sigma') = \mu_{m,1},$$

and hence, by equations $e^{F_m}_{m+1}\circ\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m = 1_{F_m}$ and (4.6), we obtain

$$\begin{aligned}
\mu_{m,1} &= e^{F_m}_{m+1}\circ\nabla_{P^{m+1}\Omega F_m}\circ(\hat\lambda_{m+1}\circ(\sigma_m\times\sigma') \vee \delta_{m+1})\circ\nu^{m,1}_{m+1} \\
&= \nabla_{F_m}\circ(\mu_{m,1} \vee e^{F_m}_{m+1}\circ\delta_{m+1})\circ\nu^{m,1}_{m+1}.
\end{aligned}$$

Thus we obtain $e^{F_m}_{m+1}\circ\delta_{m+1} = *$. Therefore, we obtain

$$\begin{aligned} e^{F_m}_{m+1}\circ\hat{\lambda} &= e^{F_m}_{m+1}\circ\nabla_{P^{m+1}\Omega F_m}\circ(\hat{\lambda}_{m+1} \vee \delta_{m+1}\circ h\circ\bar{e})\circ\hat{\nu}_{m+1} \\ &= \nabla_{F_m}\circ(e^{F_m}_{m+1}\circ\hat{\lambda}_{m+1} \vee *)\circ\hat{\nu}_{m+1} = e^{F_m}_{m+1}\circ\hat{\lambda}_{m+1}, \end{aligned}$$

and hence, by Proposition 2.4 (1) for $f = \lambda$, we proceed finally as

$$e^{F_m}_{m+1}\circ\hat{\lambda} = \mu_{m,1}\circ\{e^{F_m}_{m}\times e'\} : \hat{F}_{m+1} \to F_m.$$

It completes the proof of the lemma. □

Now we are ready to define a cone-decomposition $\{\hat{E}'_k \xrightarrow{\hat{w}'_k} \hat{F}'_{k-1} \xhookrightarrow{\hat{i}'_{k-1}} \hat{F}'_k \mid 1 \le k \le m{+}1\}$ of $P^m_m\times\Sigma\Omega A$ of length $m{+}1$ by replacing F'_1 with A in the cone-decomposition of $P^m_m\times\Sigma\Omega F'_1$. The series of cofibre sequences

$$\{E^k\Omega F_m \xrightarrow{p^{\Omega F_m}_{k-1}} P^{k-1}\Omega F_m \xhookrightarrow{\iota^{\Omega F_m}_{k-1}} P^k\Omega F_m \mid 1{\le}k{\le}m{+}1\}$$

gives a cone-decomposition of $P^{m+1}\Omega F_m$ of length $m+1$. Let D be the homotopy pushout of $\phi = \iota^{\Omega F_m}_{m,m+1}\circ pr_1$ and $\hat{\lambda}\circ\chi = \hat{\lambda}\circ(1_{P^m_m}\times\Sigma\Omega\alpha)$:

$$\begin{array}{ccc} P^m_m\times\Sigma\Omega A & \xrightarrow{\hat{\lambda}\circ\chi} & P^{m+1}\Omega F_m \\ \downarrow{\scriptstyle\phi} & & \downarrow \\ P^{m+1}\Omega F_m & \longrightarrow & D. \end{array}$$

We give a cone-decomposition of D as follows. $\hat{\lambda}\circ\hat{\iota}_m = \nabla_{P^{m+1}\Omega F_m}\circ(\hat{\lambda}_{m+1} \vee \delta_{m+1}\circ h\circ\bar{e})\circ\hat{\nu}_{m+1}\circ\hat{\iota}_m = \hat{\lambda}_{m+1}\circ\hat{\iota}_m$, we may identify the restriction of $\hat{\lambda}$ on $\hat{F}_k$ with $\hat{\lambda}_k$ and hence $\hat{\lambda}\circ\chi$ is a filtered map up to homotopy, i.e., $(\hat{\lambda}\circ\chi)|_{\hat{F}'_k} = \hat{\lambda}_k\circ\chi|_{\hat{F}'_k}$ for $1{\le}k{\le}m$. Since $\chi|_{\hat{F}'_{k-1}} = \chi|_{\hat{F}'_k}\circ\hat{i}'_{k-1}$ and $\hat{i}'_{k-1}\circ\hat{w}'_k = *$, we have

$$e^{F_m}_{k-1}\circ((\hat{\lambda}\circ\chi)|_{\hat{F}'_{k-1}}\circ\hat{w}'_k) = e^{F_m}_k\circ\hat{\lambda}_k\circ\chi|_{\hat{F}'_k}\circ\hat{i}'_{k-1}\circ\hat{w}'_k = e^{F_m}_k\circ\hat{\lambda}_k\circ\chi|_{\hat{F}'_k}\circ * = *.$$

By a standard argument of homotopy theory on a fibre sequence $E^k\Omega F_m \to P^{k-1}\Omega F_m \to F_m$, we have a lift $\kappa_k : \hat{E}'_k \to E^k\Omega F_m$ which fits in with the following commutative diagrams:

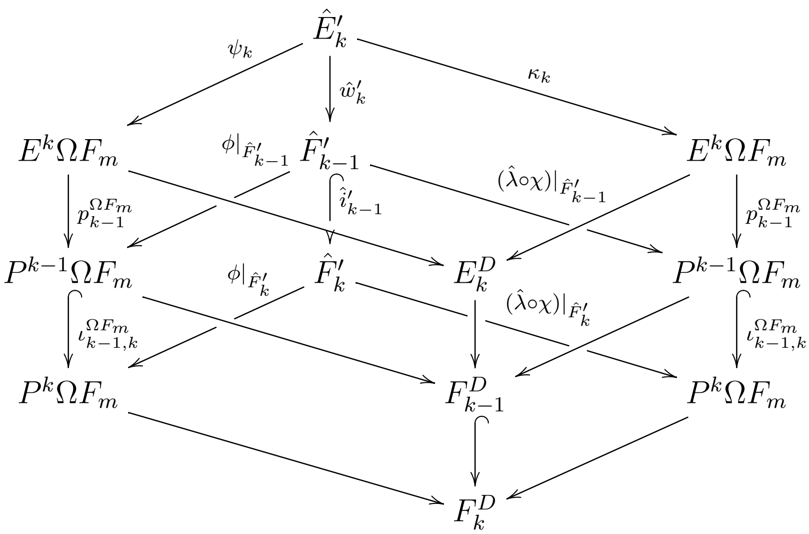
(4.7)
$$\begin{array}{ccccc} \hat{E}'_k & \xrightarrow{\hat{w}'_k} & \hat{F}'_{k-1} & \xhookrightarrow{\hat{i}'_{k-1}} & \hat{F}'_k \\ \downarrow{\scriptstyle\kappa_k} & & \downarrow{\scriptstyle\hat{\lambda}_{k-1}\circ\chi|_{\hat{F}'_{k-1}}} & & \downarrow{\scriptstyle\hat{\lambda}_k\circ\chi|_{\hat{F}'_k}} \\ E^k\Omega F_m & \xrightarrow{p^{\Omega F_m}_{k-1}} & P^{k-1}\Omega F_m & \xhookrightarrow{\iota^{\Omega F_m}_{k-1,k}} & P^k\Omega F_m \end{array} \quad (1{\le}k{\le}m),$$

(4.8)
$$\begin{array}{ccccc} \hat{E}'_{m+1} & \xrightarrow{\hat{w}'_{m+1}} & \hat{F}'_m & \xhookrightarrow{\hat{i}'_m} & \hat{F}'_{m+1} \\ \downarrow{\scriptstyle\kappa_{m+1}} & & \downarrow{\scriptstyle\hat{\lambda}_m\circ\chi|_{\hat{F}'_m}} & & \downarrow{\scriptstyle\hat{\lambda}\circ\chi} \\ E^{m+1}\Omega F_m & \xrightarrow{p^{\Omega F_m}_m} & P^m\Omega F_m & \xhookrightarrow{\iota^{\Omega F_m}_{m,m+1}} & P^{m+1}\Omega F_m \end{array} \quad (k{=}m{+}1).$$

By definition of ϕ, it is clear that there exists a map $\psi_k : \hat{E}'_k \to E^k\Omega F_m$ which fits in with the following commutative diagram:

$$
\begin{array}{ccccc}
\hat{E}'_k & \xrightarrow{\hat{w}'_k} & \hat{F}'_{k-1} & \xhookrightarrow{\hat{i}'_{k-1}} & \hat{F}'_k \\
\downarrow{\scriptstyle \psi_k} & & \downarrow{\scriptstyle \phi|_{\hat{F}'_{k-1}}} & & \downarrow{\scriptstyle \phi|_{\hat{F}'_k}} \\
E^k\Omega F_m & \xrightarrow{p^{\Omega F_m}_{k-1}} & P^{k-1}\Omega F_m & \xhookrightarrow{\iota^{\Omega F_m}_{k-1,k}} & P^k\Omega F_m.
\end{array}
\tag{4.9}
$$

Let E^D_k be a homotopy pushout of κ_k and ψ_k, and F^D_k be a homotopy pushout of $(\hat{\lambda}\circ\chi)|_{\hat{F}'_k}$ and $\phi|_{\hat{F}'_k}$, then using diagrams (4.7), (4.8) and (4.9) and using the universal property of the homotopy pushouts, we obtain the following commutative diagram such that the front column $E^D_k \to F^D_{k-1} \to F^D_k$ is a cofibre sequence:



Thus we obtain a cone-decomposition $\{E^D_k \to F^D_{k-1} \hookrightarrow F^D_k \mid 1 \le k \le m{+}1\}$ of D of length $m{+}1$, which immediately implies the following inequalities.

$$\mathrm{cat}(D) \le \mathrm{Cat}(D) \le m{+}1.$$

The homotopy pushout of top and bottom rows in (4.4) are $G\cup_\psi G\times CA$. Also, since dimensions of F_m, F_1 and A are less than or equal to ℓ, all composition of columns in (4.4) are homotopy equivalences. Thus, we obtain a composite map $D \to G \cup_\psi G\times CA \simeq E \to D$ as a homotopy equivalence (see [4, Lemma 16.24], for example). Thus D dominates E and we obtain

$$\mathrm{cat}(E) \le \mathrm{cat}(D) \le \mathrm{Cat}(D) \le m{+}1.$$

5. L-S Category of SO(10)

In this section, we determine $\mathrm{cat}(\mathbf{SO}(10))$ and prove Theorem 5.1.

To give a lower bound of $\mathrm{cat}(\mathbf{SO}(10))$, let us recall the algebra structure of the well-known cohomology algebra $H^*(\mathbf{SO}(10);\mathbb{F}_2)$ as described below:

$$H^*(\mathbf{SO}(10);\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_3,x_5,x_7,x_9]/(x_1^{16},x_3^4,x_5^2,x_7^2,x_9^2),$$

where x_k is a generator in dimension k. Then by Theorem 1.1, we obtain

$$21 = \mathrm{cup}(\mathbf{SO}(10);\mathbb{F}_2) \le \mathrm{cat}(\mathbf{SO}(10)). \tag{5.1}$$

On the other hand, to give the upper bound using Theorem 1.2, firstly we recall the cone-decomposition of $\mathbf{Spin}(7)$ in [10] as follows:

$$* \subset F'_1 = \Sigma\mathbb{C}\mathrm{P}^3 \subset F'_2 \subset F'_3 \subset F'_4 \subset F'_5 \simeq \mathbf{Spin}(7).$$

In [11], the cone-decomposition of $\mathbf{SO}(9)$ is given by using the above filtration F'_i of $\mathbf{Spin}(7)$ together with the principal bundle $\mathbf{Spin}(7) \hookrightarrow \mathbf{SO}(9) \to \mathbb{R}\mathrm{P}^{15}$: let e^k be a k-cell in $\mathbf{SO}(9)$ corresponding to the k-cell in $\mathbb{R}\mathrm{P}^{15}$. The cone-decomposition $\{F_i\}$ of $\mathbf{SO}(9)$ introduced in [11] is as follows.

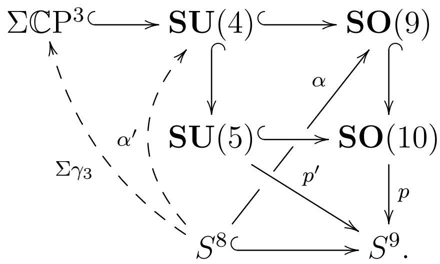
$$\begin{aligned}
F_0 &= \{*\}\\
&\ \vdots \qquad \ddots\\
F_j &= F'_j \cup (e^1{\times}F'_{j-1}) \cup \cdots \cup (e^{j-1}{\times}F'_1) \cup e^j\\
&\ \vdots \qquad\qquad \ddots\\
F_5 &= F'_5 \cup (e^1{\times}F'_4) \cup (e^2{\times}F'_3) \cup (e^3{\times}F'_2) \cup (e^4{\times}F'_1) \cup e^5\\
&\ \vdots \qquad\qquad\qquad \ddots\\
F_{i+5} &= F'_5 \cup (e^1{\times}F'_5) \cup \cdots \cup (e^i{\times}F'_5) \cup (e^{i+1}{\times}F'_4) \cup \cdots \cup (e^{i+4}{\times}F'_1) \cup e^{i+5}\\
&\ \vdots \qquad\qquad\qquad \vdots\\
F_{15} &= F'_5 \cup (e^1{\times}F'_5) \cup \cdots \cup (e^{10}{\times}F'_5) \cup (e^{11}{\times}F'_4) \cup \cdots \cup (e^{14}{\times}F'_1) \cup e^{15}\\
&\ \vdots \qquad\qquad\qquad \vdots\\
F_{15+j} &= F'_5 \cup (e^1{\times}F'_5) \cup \cdots \cup (e^{10+j}{\times}F'_5) \cup (e^{11+j}{\times}F'_4) \cup \cdots \cup (e^{15}{\times}F'_{5-j})\\
&\ \vdots \qquad\qquad\qquad \vdots\\
F_{20} &= F'_5 \cup (e^1{\times}F'_5) \cup \cdots \cup (e^{15}{\times}F'_5) \simeq \mathbf{SO}(9)
\end{aligned}$$

where $0 \le i \le 10$ and $0 \le j \le 5$, which is given with a series of cofibre sequences $\{K_i \to F_{i-1} \to F_i \mid 1{\le}i{\le}20\}$.

Secondly, a cofibre sequence $S^{20} \to F'_4 \hookrightarrow F'_4 \cup e^{21}$ $(= F'_5 \simeq \mathbf{Spin}(9))$ in [10] induces a cofibre sequence $K_{20} = S^{14} * S^{20} = S^{35} \to F_{19} \hookrightarrow F_{20}$.

Thirdly, since $\mu'|_{F'_i\times F'_1}$ is compressible into F'_{i+1} for $1 \le i < 5$ by the proof of [11, Theorem 2.9], $\mu|_{F_i\times F'_1}$ is compressible into F_{i+1} for $1 \le i < 20$, where μ and μ' are multiplications of $\mathbf{SO}(9)$ and $\mathbf{Spin}(7)$, respectively.

Fourthly, let us consider two principal bundles $p : \mathbf{SO}(10) \to S^9$ and $p' : \mathbf{SU}(5) \to S^9$, together with the following commutative diagram:



The map $\alpha : S^8 \to \mathbf{SO}(9)$ in the above diagram is a characteristic map of $p : \mathbf{SO}(10) \to S^9$. By Steenrod [16], α is homotopic in $\mathbf{SO}(9)$ to a map $\alpha' : S^8 \to \mathbf{SU}(4)$ the characteristic map of $p' : \mathbf{SU}(5) \to S^9$. Further by Yokota [18], the suspension $\Sigma\gamma_3 : S^8 \to \Sigma\mathbb{C}\mathrm{P}^3$ of the canonical projection $\gamma_3 : S^7 \to \mathbb{C}\mathrm{P}^3$ is the attaching map of the top cell of $\Sigma\mathbb{C}\mathrm{P}^4 \subset \mathbf{SU}(5)$, which is homotopic to α'. Therefore, the characteristic map α is compressible into $\Sigma\mathbb{C}\mathrm{P}^3 \subset F_1$. Since α is homotopic to a suspension map to $\Sigma\mathbb{C}\mathrm{P}^3$ in $\mathbf{SO}(9)$, and hence we have $H_1(\alpha) = 0 \in \pi_8(\Omega\Sigma\mathbb{C}\mathrm{P}^3 {*} \Omega\Sigma\mathbb{C}\mathrm{P}^3)$ when α is regarded to be a map to $\Sigma\mathbb{C}\mathrm{P}^3$.

Thus, finally by Theorem 1.2 with $F'_1 = \Sigma\mathbb{C}\mathrm{P}^3$, we obtain

$$\operatorname{cat}(\mathbf{SO}(10)) \leq 20+1 = 21. \tag{5.2}$$

Combining (5.2) with (5.1), we obtain our desired result.

Theorem 5.1. $\operatorname{cat}(\mathbf{SO}(10)) = 21 = \operatorname{cup}(\mathbf{SO}(10); \mathbb{F}_2)$.

Acknowledgements. We would like to express our gratitude to the referee for his/her kind comments and suggestions which helped us to improve the presentation of this paper. This research was supported by Grant-in-Aid for Scientific Research (B) #22340014 and (A) #23244008 and for Exploratory Research #24654013 from Japan Society for the Promotion of Science.

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