Splitting off rational parts in homotopy types

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1 Rational case

A graded module H over a field K is called a Hopf algebra if there are homomorphisms of K-modules

$$\phi : H \otimes H \to H, \quad \eta : K \to H,$$

$$\psi : H \to H \otimes H, \quad \epsilon : H \to K,$$

such that

(1) (H, ϕ) is an algebra with two-sided unit $\eta(1)$,

(2) (H^*, ψ^*) is an algebra with two-sided unit $\epsilon^*(1)$,

(3) ϵ and η^* are homomophisms of algebras and

(4) ψ and ϕ^* are homomorphisms of algebras.

(Borel [Bo67]) A connected commutative associative Hopf algebra H of finite type over $\mathbb{Q}$ is isomorphic as algebra to the tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd:

$$H \cong P[x_1, x_2, \cdots] \otimes E(y_1, y_2, \cdots).$$

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The resulting formula can be rewritten as

$$H \cong \bigotimes_{n=1}^{\infty} A[x_n]$$

where x_n is a homogeneous generator of dimension $d_n \geq 1$ and $A[x]$ is defined as follows: $A[x] = P[x]$ if d_n is even and $A[S] = E(S)$ if d_n is odd.

A space X is called a Hopf space if there is a map

$$\mu : X \times X \longrightarrow X$$

such that μ has two-sided homotopy unit.

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(Scheerer [Sch85]) If a rational space X_0 is a Hopf space, then X_0 has the homotopy type of a generalized Eilenberg-Mac Lane space:

$$X_0 \simeq \bigoplus_{n=1}^{\infty} K(\pi_n(X_0); n)$$

where $\bigoplus$ denotes the weak product:

$$\bigoplus_{\lambda \in \Lambda} X_\lambda = \left\{ (x_\lambda) \in \prod_{\lambda \in \Lambda} X_\lambda \;\middle|\; \begin{array}{l} x_\lambda = * \text{ except for} \\ \text{finitely many } \lambda \end{array} \right\}$$

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Can it happen only for a Hopf space?

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(Oprea [Op86]) If X_0 is a rational G-space of finite type, then X_0 has the homotopy type of a generalized Eilenberg-Mac Lane space.

(Aguadé [Ag87]) If X_0 is a rational T-space of finite type, then X_0 has the homotopy type of a generalized Eilenberg-Mac Lane space.

Defn 1.1

$$G(X,Y) = \left\{ [f] \in [X,Y] \,\middle|\, \begin{array}{l} \exists_{F:X\times Y\to Y} \ s.t.\ F \\ has\ axes\ f\ and\ 1_Y \end{array} \right\}$$

Defn 1.2

(1) (Gottlieb [Go69]) A space X is a G-space iff $\pi_n(X) = G(S^n, X)$ for all $n \geq 1$.

(2) (Aguadé [Ag87], Woo-Yoon [WY95]) A space X is a T-space iff $[\Sigma A, X] = G(\Sigma A, X)$ for any space A.

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What happens in non-rational case?

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(L. Fuchs) *Any abelian group A is a direct sum of a divisible group and a reduced group:*

$$A \cong \textit{(disivible part)} \oplus \textit{(reduced part)}$$

2 Non-rational case

Let $\overline{\rho} : [S^n_{\mathbb{Q}}, X] \to H_n(X)$ be a homomorphism defined by $\overline{\rho}(\alpha) = \alpha_*([S^n]\otimes 1)$, where we regard $H_n(S^n_{\mathbb{Q}}) = H_n(S^n)\otimes\mathbb{Q}$.

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Thm 2.1 *Let R be a finite or an infinite dimensional $\mathbb{Q}$-vector space. If $R \subset \overline{\rho}(G(S^n_{\mathbb{Q}}, X)) \subseteq H_n(X)$ $(n \geq 2)$, then we have*

$$X \simeq Y \times K(R, n).$$

Cor 2.1.1 *Let R be a finite or an infinite dimensional $\mathbb{Q}$-vector space. Let X be an $(n-1)$-connected T-space with $R \subseteq H_n(X)$ $(n \geq 2)$. Then X decomposes as*

$$X \simeq Y \times K(R, n) \quad \textit{for a } T\textit{-space } Y.$$

3 Rational case revisited

Thm 3.1 *Let* $R = \bigoplus_{\lambda\in\Lambda} \mathbb{Q}$ *be a finite or an infinite dimensional* $\mathbb{Q}$*-vector space. If a rational space* X_0 *is an* $(n-1)$*-connected* G*-space with* $H_n(X_0) \supseteq R$ *for* $n \geq 2$*, then* X_0 *decomposes as*

$$X_0 \simeq Y_0 \times K(R, n),$$

where Y_0 *is an rational* G*-space.*

Cor 3.1.1 *Let* X *be a* 0*-connected CW complex with rationalization* $X_{\mathbb{Q}}$*, then the following conditions are equivalent:*

(1) $X_{\mathbb{Q}}$ *is a* G*-space.*

(2) $X_{\mathbb{Q}}$ *is a* T*-space.*

(3) $X_{\mathbb{Q}}$ *is a Hopf space.*

(4) *Every* k*-invariant of* X *is of finite order.*

Cor 3.1.2 *If the rationalization $X_{\mathbb{Q}}$ of a 0-connected virtually nilpotent space X is a G-space, then $X_{\mathbb{Q}}$ has the homotopy type of a weak product of Eilenberg-Mac Lane spaces:*

$$X_{\mathbb{Q}} \simeq \bigoplus_{n=1}^{\infty} K(\pi_n(X_{\mathbb{Q}}); n)$$

Cor 3.1.3 *Let X be a 1-connected rational G-space. Then $X_{\mathbb{Q}}$ is a Hopf space and the Hopf algebra $H_*(X;\mathbb{Q})$ is isomorphic as co-algebra with a tensor product of a polynomial algebra on even dimensional generators and an exterior algebra on odd, where generators may be infinitely many:*

$$H_*(X;\mathbb{Q}) \cong \bigotimes_{\lambda\in\Lambda} A[x_\lambda], \quad \text{as coalgebras,}$$

where Λ denotes an index set of homogeneous generators.

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