L-S category and a categorical sequence

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‘Cone Decomposition’ + ‘Higher Hopf invariant’

||

‘Categorical Sequence’

L-S category

Definition (Lusternik-Schnirelmann)

$$\mathrm{cat}(M) = \mathrm{Min}\left\{ m \geq 0 \;\middle|\; \begin{array}{l} \exists\{A_0, ..., A_m \,;\text{ closed in } M\} \\ M = \bigcup_{i=0}^{m} A_i, \begin{array}{l}\text{where each } A_i \text{ is} \\ \text{contractible in } M.\end{array} \end{array} \right\}$$

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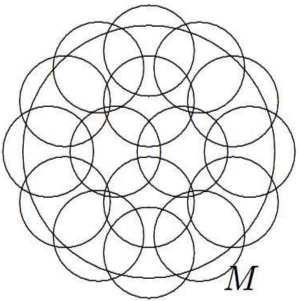

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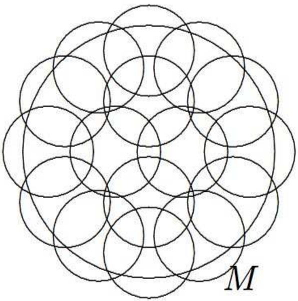

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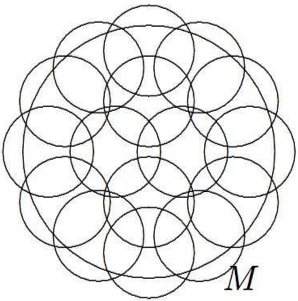

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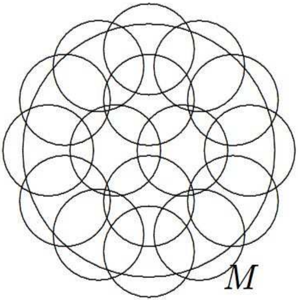

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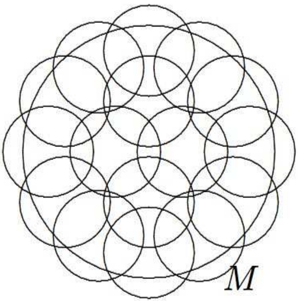

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this definition gives only an upper bound for $\mathrm{cat}(M)$.

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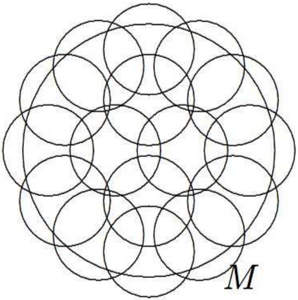

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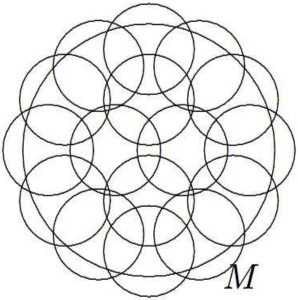

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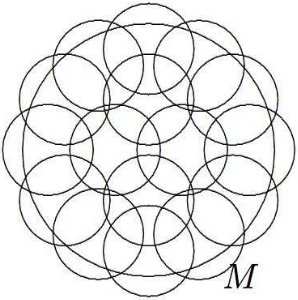

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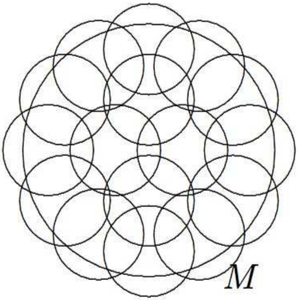

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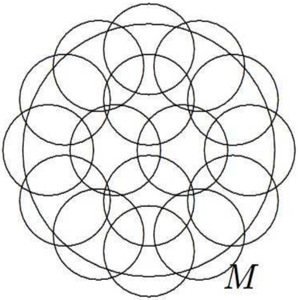

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Fact

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For a map f from S^q to a space X with $\mathrm{cat}(X) = m$,

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Theorem (Stasheff)

For any space X, the space of all loops at the base point of X admits a natural A_∞-structure,

associated with projective spaces $P^m\Omega(X)$ and natural maps

$$e^X_m : P^m\Omega(X) \longrightarrow X, \; m \geq 0.$$

A_∞-structure on a space

Theorem (Stasheff)

For any space X, the space of all loops at the base point of X admits a natural A_∞-structure,

associated with projective spaces $P^m\Omega(X)$ and natural maps

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Using this criterion, we can now determine more L-S categories...

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Projective spaces and a higher Hopf invariant

Thus the difference $d_m^{\sigma(X)}(f)$ has a unique lift

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If an A_m-structure of an A_m-space S can be extended to an A_{m+1}-structure, then there must be an element $[f] \in [S, P^m(S)]$ with $H_m(f) = 1$.

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By combining a cone decomposition with a higher Hopf invariant, Kikuchi and I obtain a categorical sequence, and eventually get a better upper bound of L-S category.

Proposition (Kikuchi, I)

$$\mathrm{catseq}(SO(10)) \leq 21.$$

On the other hand, we know

$$H^*(SO(10);\mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_3, x_5, x_7, x_9]/(x_1^{16}, x_3^4, x_5^2, x_7^2, x_9^2),$$

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Outline of the proof of $\mathrm{catseq}(\mathrm{SO}(10)) = 21$

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Outline of the proof of $\mathrm{catseq}(SO(10)) = 21$

Using $H_1(\alpha) = 0$, we obtain that the sequence $\{E_i\,;\ 0 \leq i \leq 21\}$ of subspaces of $SO(10)$ is categorical and hence we have

$$\mathrm{catseq}(SO(10)) \leq 21.$$

While we do not know whether the above sequence $\{E_i\,;\ 0 \leq i \leq 21\}$ itself is a cone decomposition of $SO(10)$ or not, Using $H_1(\alpha) = 0$, we obtain that the sequence $\{E_i\,;\, 0 \le i \le 21\}$ of subspaces of $SO(10)$ is categorical and hence we have

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Problem

Do the following three invariants for $\mathrm{SO}(n)$ *coincide with each other? i.e.,*

$$\mathrm{cup}(\mathrm{SO}(n)) =?\, \mathrm{cat}(\mathrm{SO}(n)) =?\, \mathrm{Cat}(\mathrm{SO}(n))$$

Thank you.