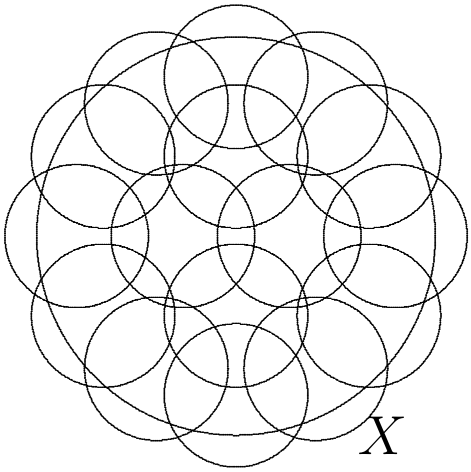
L-S CATEGORY OF A SPHERE-BUNDLE OVER A SPHERE

AMS-IMS-SIAM Joint Summer Research Conference
Lusternik-Schnirelmann Category in the New Millennium
Sunday 29th July – Thursday 2nd August, 2001

Norio IWASE

1 What is the L-S category of a space X



Definition 1.1

$$\operatorname{cat}(X) = \operatorname{Min}\left\{ m \geq 0 \;\middle|\; \begin{array}{l} \exists\{U_0, ..., U_m\,;\ \text{open in } X\} \\ X = \bigcup_{i=0}^{m} U_i,\ \text{each } U_i \text{ is contractible } \underline{in}\ X \end{array} \right\}$$

Theorem 1.2 (Lusternik-Schnirelmann)

$$\#\{\textit{critical points of a } C^\infty\textit{-map } f : M \to \mathbb{R}\} > \operatorname{cat} M.$$

But generally speaking, a simple definition does not suggest a simple way of calculation.

2 Ganea's problems

Problems [T. Ganea, 1971, (15 problems)]

[1] Compute $\operatorname{cat} M$ for a closed manifold M.

[2] $\operatorname{cat} X \times S^n = \operatorname{cat} X + 1$?

[4] Let $S^r \hookrightarrow E \to S^{t+1}$ be a bundle. Describe $\operatorname{cat} E$ in terms of homotopy invariants of the characteristic map of the bundle. $(E = S^r \cup_\Psi D^{t+1} \times S^r,\ \Psi : S^t \times S^r \to S^r)$

...

[O] For a closed manifold M, $\operatorname{cat} M > \operatorname{cat}(M - \{x\})$?

Remark 2.1 *If [O] is true, then so is [2] (Ganea's conjecture in L-S category) for closed manifolds.*

Remark 2.2 *For* $E = S^r \cup_\Psi D^{t+1} \times S^r,\ E - \{x\} \simeq S^r \cup_\alpha D^{t+1}$.

2.1 Hopf invariants

Classically, a Hopf invariant is defined to detect the existence of a multiplicative structure with unit on a sphere S^{n-1}:

$$H : [S^{2n-1}, S^n] \to \mathbb{Z}, \quad S^{2n-1} = S^{n-1} * S^{n-1}, S^n = \Sigma S^{n-1},$$

where $\Sigma X = \{-1\} * X * \{1\}$ and $A_1 * A_2 = \{ta_1 + (1-t)a_2 \mid a_i \in A_i, t \in [0,1]\}$. While the homotopy set $[\Sigma X, W]$ has a natural group structure, the induced map $f^* : [\Sigma B, W] \to [\Sigma A, W]$ from a map $f : \Sigma A \to \Sigma B$ is *not* a homomorphism, in general:

$$f^*(\alpha + \beta) \neq f^*(\alpha) + f^*(\beta), \quad \text{for } \alpha, \beta \in [\Sigma B, W].$$

The difference is given by the composition $[\alpha, \beta] \circ h_2(f)$ of the Whitehead product $[\alpha, \beta]$ and the Hopf invariant $H_1(f)$.

[Berstein-Hilton] $H_1(f)$ determines $\mathrm{cat}\,(S^r \cup_f D^{t+1})$.

2.2 (L-S category and Higher Hopf invariants)

$\mathrm{cat}\, X \le m \overset{\text{[Whitehead]}}{\underset{(1957)}{\Longleftrightarrow}} \Delta^{m+1} : X \longrightarrow \prod^{m+1} X$

h

$^{\exists}s$

(fat wedge) $\mathrm{T}^{m+1} X$

[Berstein-Hilton] (1968)

$H^s_m(f) \in \pi_{n+1}(\prod^{m+1} X, \mathrm{T}^{m+1} X; A)$, $(f : M(A,n) \to X)$

[Berstein-Hilton] (1968)

$\mathrm{cat}\, S^r \cup_f D^{t+1} = 2 \iff H_1(f) = 0$, $(f : S^t \to S^r)$

$\mathrm{cat}\, X \le m \overset{\text{[Ginsburg]}}{\underset{(1962)}{\Longrightarrow}}$ $^{\exists}\{E^r_{**}, d^r\}$ Bar spectral sequence for $H_*(\Omega X)$ such that $d^r = 0$ for $r > m$.

[Ganea] (1967)

[Gilbert] (1968)

[Sakai] (2000)

$E^{m+1}(\Omega X) \hookrightarrow E^{\infty}(\Omega X) \simeq *$

$p^{\Omega X}_{m+1}$ PB

$X \xrightarrow{\ ^{\exists}\sigma\ } P^m(\Omega X) \overset{e^X_m}{\hookrightarrow} P^{\infty}(\Omega X) \simeq X$

1_X

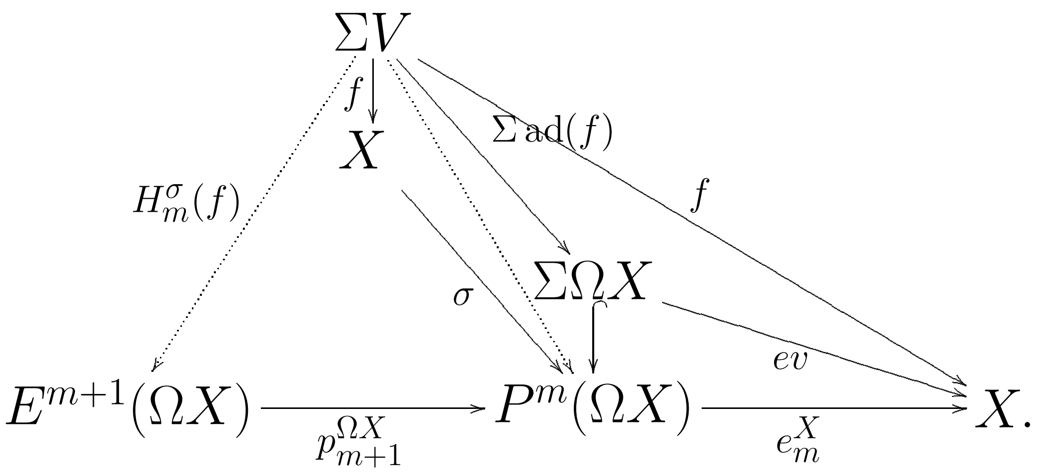
I (1998)

$H^S_m(f) = \{H^{\sigma}_m(f) \mid \sigma\} \subset [\Sigma V, E^{m+1}(\Omega X)]$, $(f : \Sigma V \to X)$

I (1998) $\searrow$ Counter Examples to [2]

Definition 2.3 *For $f \in [\Sigma V, X]$, we have*

$$e_m^X \circ \Sigma\,\mathrm{ad}(f) = ev \circ \Sigma\,\mathrm{ad}(f) = f = 1_X \circ f = e_m^X \circ \sigma \circ f,$$



Thus the difference $d_m^\sigma(f) = \sigma \circ f - \Sigma\,\mathrm{ad}(f)$ defines $H_m^\sigma(f)$ by

$p_{m+1}^{\Omega X} \circ H_m^\sigma(f) = d_m^\sigma(f)$ and $\mathcal{H}_m^\sigma(f) = \Sigma^\infty H_m^\sigma(f)$.

$$H_m^S(f) = \left\{ H_m^\sigma(f) \,\middle|\, \begin{array}{l} \sigma \text{ is a structure} \\ \text{for cat}\, X = m \end{array} \right\} \subset [\Sigma V, E^{m+1}(\Omega X)]$$

$$\mathcal{H}_m^S(f) = \left\{ \mathcal{H}_m^\sigma(f) \,\middle|\, \begin{array}{l} \sigma \text{ is a structure} \\ \text{for cat}\, X = m \end{array} \right\} \subset \{\Sigma V, E^{m+1}(\Omega X)\}$$

Theorem 2.4 *$H_m^S : [S^{n(m+1)-1}, P^m(S^{n-1})] \to \mathbb{Z}$ detects the A_m-structure of the multiplication of S^{n-1} (n=1,2,4,8), where $P^m(S^{n-1})$ is the m-th projective space.*

3 LS category of a sphere-bundle over a sphere

Let M be the total space of a sphere-bundle over a sphere:

$$S^r \hookrightarrow M \to S^{t+1} \quad \text{with structure group } G,$$

$$M = S^r \cup_\Psi D^{t+1}{\times}S^r, \quad \Psi = \mu{\circ}(\bar{\alpha}\times 1) : S^t{\times}S^r \longrightarrow S^r,$$

where $\mu : G{\times}S^r \to S^r$ is the action of G on the fibre S^r and $\bar{\alpha} : S^t \to G$ is the characteristic map of the bundle. Thus

$M = Q \cup_\psi D^{r+t+1},\ Q = M - \{x\} \simeq S^r \cup_\alpha D^{t+1},$

$\alpha = \Psi|_{S^t\times\{*\}},\ \psi \simeq [\iota_r, \chi_{t+1}]^r$ a relative Whitehead product.

- ($\dim M \leq 3$) $\operatorname{cat} M$ is determined completely by Singhof, Montejano, Gomez-Gonzalez and Rudyak.

- ($\operatorname{cat} Q \leq 1$) $\operatorname{cat} M$ is well-known by cup-length arguments.

4 An answer to Problem 4

Theorem 4.1 *For $E = S^r \cup_\alpha D^t \cup_\psi D^{r+t+1}$ a closed manifold with $H_1(\alpha) \neq 0$, we have "$H_2^S(\psi) \not\ni 0 \iff \mathrm{cat}\, E = 3$":*

<table>
<tr><th colspan="3">Conditions</th><th colspan="4">L-S category</th></tr>
<tr><th>r</th><th>t</th><th>α</th><th>$Q\times S^n$</th><th>Q</th><th>E</th><th>$E\times S^n$</th></tr>
<tr><td rowspan="5">$r=1$</td><td>$t=0$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t=1$</td><td>$\alpha=\pm1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha=0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>otherwise</td><td>3</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$t>1$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="6">$r>1$</td><td>$t<r$</td><td></td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="2">$t=r$</td><td>$\alpha=\pm1$</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>$\alpha\neq\pm1$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td rowspan="3">$t>r$</td><td>$H_1(\alpha)=0$</td><td>2</td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>$H_1(\alpha)\neq0$
$H_2^S(\psi)\ni0$</td><td rowspan="2">3 or 2
(1)</td><td rowspan="2">2</td><td>2</td><td>3</td></tr>
<tr><td>$H_1(\alpha)\neq0$
$H_2^S(\psi)\not\ni0$</td><td>3</td><td>3 or 4
(2)</td></tr>
</table>

$$(1)\ \begin{cases} \Sigma^n H_1(\alpha) = 0 \implies \mathrm{cat}\, Q\times S^n = 2, \\ \Sigma^{n+1} H_1(\alpha) \neq 0 \implies \mathrm{cat}\, Q\times S^n = 3. \end{cases}$$

$$(2)\ \begin{cases} \Sigma^{r+n} H_1(\alpha) = 0 \implies \mathrm{cat}\, E\times S^n = 3, \\ \Sigma^{r+n+1} h_2(\alpha) \neq 0 \implies \mathrm{cat}\, E\times S^n = 4. \end{cases}$$

5 Manifold counter examples

Let us consider the Hopf (principal) fibrations

$$S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2 \quad \text{with structure group } U(1) \approx S^1,$$

$$S^3 \hookrightarrow S^7 \xrightarrow{\nu} S^4 \quad \text{with structure group } Sp(1) \approx S^3.$$

By taking orbit space of the action of $U(1) \subset Sp(1)$, we obtain

$$S^2 \hookrightarrow \mathbb{C}P^3 \xrightarrow{\pi} S^4 \quad \text{with structure group } Sp(1).$$

Definition 5.1 *For $\beta \in \pi_t(S^3) = [S^t, S^r]$, we have*

$$\begin{array}{ccc} E(\beta) & \longrightarrow & \mathbb{C}P^3 \\ \downarrow q & \textit{Pull-Back} & \downarrow \pi \\ S^{t+1} & \xrightarrow{\Sigma\beta} & S^4 \end{array}$$

$E(\beta) = Q(\beta) \cup_{\psi(\beta)} D^{t+3}, Q(\beta) = E(\beta) - \{x\} \simeq S^2 \cup_{\eta\circ\beta} D^{t+1}$.

If $H_1(\beta) = 0$, then $H_1(\eta\circ\beta) = \beta$, and hence we have

$$\mathrm{cat}(Q(\beta)) = 2 \iff \beta \neq 0.$$

Let p be an odd prime. Then $\alpha_1(3)\circ\alpha_2(2p) : S^{2p} \to S^3$ satisfies $H_1(\alpha_1(3)\circ\alpha_2(2p)) = 0$. Using results of Toda and Oka on p-primary component of $\pi_*^S(S^0)$, we obtain the following lemma.

Lemma 5.2 *The set* $\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2p)))$ *gives a map* $\pm\Sigma^3(\alpha_1(3)\circ\alpha_2(2p))$ *composed with an inclusion map.*

Using this, we obtain the following results.

Theorem 5.3 *There is a closed manifold* M *such that* $\mathrm{cat}\,(M \times S^n) = \mathrm{cat}\,M$, *for* $n \geq 1$.

Theorem 5.4 *There is a closed manifold* N_p *for each odd prime* $p \geq 5$ *such that* $\mathrm{cat}\,N_p = \mathrm{cat}\,(N_p - \{x\})$.

Remark 5.5 *Pascal Lambrecht, Don Stanley and Lucile Vandembroucq have also obtained manifolds which satisfy the same property as* N_p *in Theorem 5.4 does.*

6 Outline of the proof of Lemma 5.2

By a concrete homotopy-theoretical observation, we can show

Proposition 6.1 $\exists_{H_2^{SS}(\psi(-))\subset H_2^S(\psi(-))}$ *such that*

(1) $\beta\circ\gamma = 0 \implies (\Sigma^2\gamma)^* H_2^{SS}(\psi(\beta)) = \{0\}$ *for all* β, γ.

(2) $\ell\beta = 0 \implies \ell H_2^{SS}(\psi(\beta)) = \{0\}$ *for all* β.

For the dimensional reasons, we have (for some $a, b \in \mathbb{Z}$)

$$\Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{a\alpha_2(6) + b\alpha_1(2p+4)\}$$

with $a = 1$ by a result of Boardman-Steer, and hence

$$\Sigma E(\Omega\alpha_1\widehat{(4p-2)_0})_*\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2p)))$$

$$= \pm 2\alpha_1(4p+1)^*\Sigma_* H_2^{SS}(\psi(\alpha_2(3))) = \{\pm\alpha_1(6)\circ\alpha_2(2p+3)\}.$$

Then the set $\Sigma_* H_2^{SS}(\psi(\alpha_1(3)\circ\alpha_2(2p)))$ is described as

$$\{\pm\alpha_1(6)\circ\alpha_2(2p+3) + x(\iota_1\circ\alpha_1(4p+1) + \iota_1\circ\alpha_2(2p+3))\},$$

for some $x \in \mathbb{Z}$. By a result of Oka, we know $\alpha_1 \circ \beta_1 \neq 0$ stably, while $\alpha_2 \circ \beta_1 = 0$ stably. Thus $x = 0$ and we obtain the lemma.

7 Proofs of Theorems 5.3 and 5.4

Let $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(6)$ with $p = 3$. Then $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(6) \neq 0$, $\Sigma^2 H_1(\alpha) \neq 0$ and $\Sigma^4 H_1(\alpha) = 0$ by Toda. Let M_3 be the S^2-bundle over S^{14} induced by $\Sigma(\alpha_1(3) \circ \alpha_2(6)) : S^{14} \to S^4$ from the S^2-bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$. Lemma 5.2 implies that $\mathrm{cat}(M_3 \times S^n) = \mathrm{cat}\, M_3 = 3$ for $n \geq 2$.

Let $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_2(2p)$ with p odd ≥ 5. Then $H_1(\alpha) = \alpha_1(3) \circ \alpha_2(2p) \neq 0$ and $\Sigma^2 H_1(\alpha) = 0$ by Toda. Let N_p be the S^2-bundle over S^{6p-4} induced by $\Sigma(\alpha_1(3) \circ \alpha_2(2p)) : S^{6p-4} \to S^4$ from the S^2-bundle $\mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$. Lemma 5.2 implies that $\mathrm{cat}\, N_p = \mathrm{cat}(N_p - \{x\}) = 2$.

8 A new answer to Problem 4

Theorem 8.1 *For E an S^r-bundle over S^{t+1} with $H_1(\alpha) \neq 0$, we have "$H_2^S(\psi) \ni 0 \iff \Sigma^r H_1(\alpha) = 0$" and*

Conditions			L-S category			
r	t	α	$Q \times S^n$	Q	E	$E \times S^n$
$r = 1$	$t = 0$		2	1	2	3
	$t = 1$	$\alpha = \pm 1$	1	0	1	2
		$\alpha = 0$	2	1	2	3
		otherwise	3	2	3	4
	$t > 1$		2	1	2	3
$r > 1$	$t < r$		2	1	2	3
	$t = r$	$\alpha = \pm 1$	1	0	1	2
		$\alpha \neq \pm 1$	2	1	2	3
	$t > r$	$H_1(\alpha) = 0$	2	1	2	3
		$H_1(\alpha) \neq 0$ $\Sigma^r H_1(\alpha) = 0$	3 or 2 (1)	2	2	3
		$\Sigma^r H_1(\alpha) \neq 0$			3	3 or 4 (2)

$$(1) \begin{cases} \Sigma^{n+1} H_1(\alpha) \neq 0 \implies \operatorname{cat} Q \times S^n = 3, \\ \Sigma^n H_1(\alpha) = 0 \implies \operatorname{cat} Q \times S^n = 2. \end{cases}$$

$$(2) \begin{cases} \Sigma^{r+n} H_1(\alpha) = 0 \implies \operatorname{cat} E \times S^n = 3, \\ \Sigma^{r+n+1} h_2(\alpha) \neq 0 \implies \operatorname{cat} E \times S^n = 4. \end{cases}$$

9 Outline of the proof of Theorem 8.1

Since E is a bundle, we obtain that $H_2^S(\psi) \ni 1_{S^{r-1}} * H_1(\alpha) = \pm\Sigma^r H_1(\alpha)$. If $H_2^S(\psi) \ni H_2^\sigma(\psi) = 0$ for some σ, then it follows that $1_{S^{r-1}} * H_1(\alpha)$ is homotopic to a Whitehead product $[\iota_r, \delta]$ where $\iota_r : S^r \hookrightarrow \Sigma\Omega S^r \subset P^2(\Omega S^r)$ is the bottom-cell inclusion.

The key idea to proceed is obtained by looking at the first differential $d_1 : H_*(\Omega S^r \wedge \Omega S^r \wedge \Omega S^r) \to H_*(\Omega S^r \wedge \Omega S^r)$ of the Bar spectral sequence by Ginsburg. We observe that the subspace $S^{r-1} * \Omega S^r * \Omega S^r \subset \Omega S^r * \Omega S^r * \Omega S^r$ which contains the image of $1_{S^{r-1}} * H_1(\alpha)$ is a retract of $P^2(\Omega S^r)/\Sigma\Omega(S^r)$. Here, the Whitehead product $[\iota_r, \delta]$ vanishes after composing the retraction, since $S^r \subset \Sigma\Omega(S^r)$. Thus $\Sigma^r H_1(\alpha)$ must be trivial. The converse is clear by $H_2^S(\psi) \ni \pm\Sigma^r H_1(\alpha) = 0$.