Ganea's Problems and Their Localised Versions

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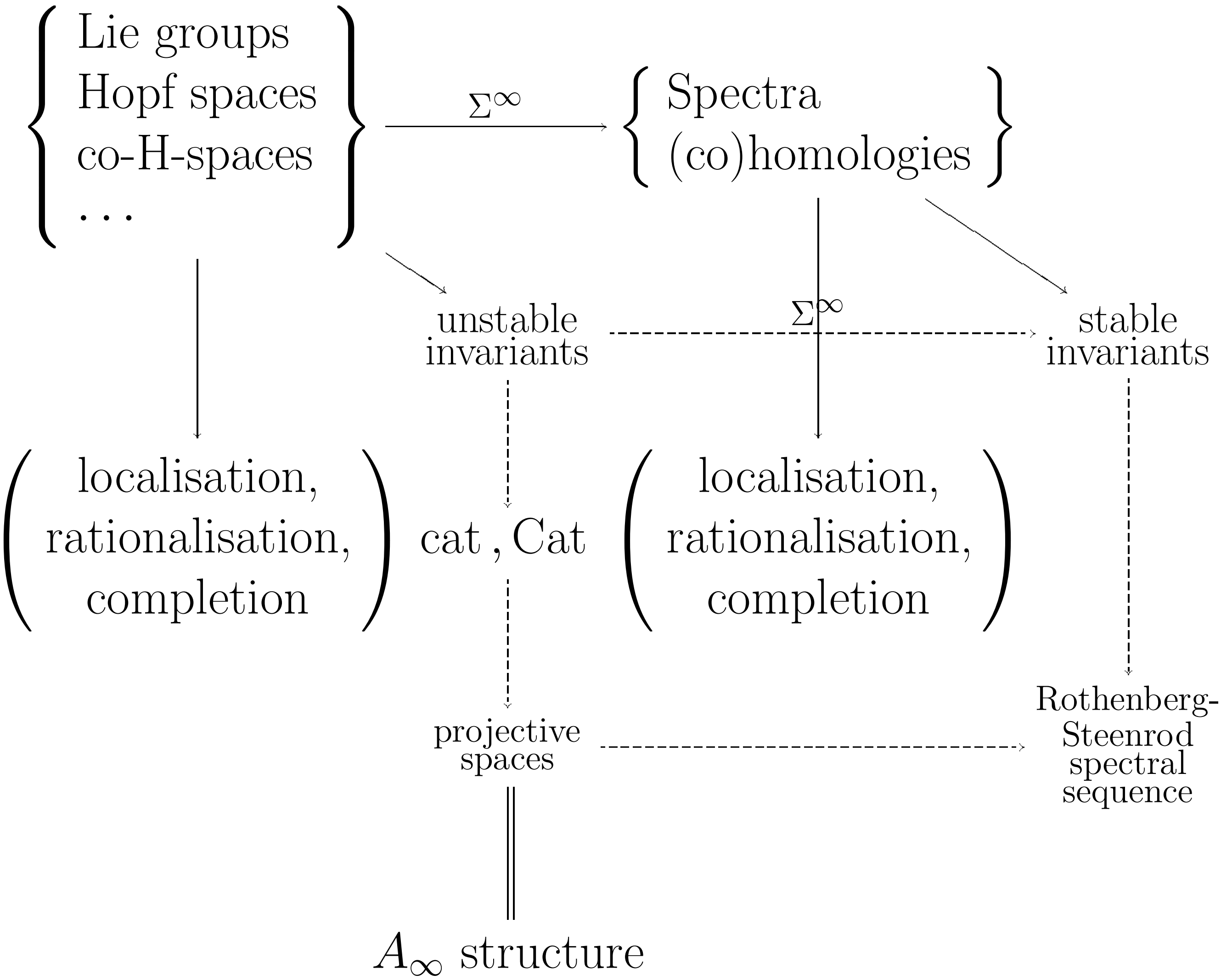
unstable theory

stable theory



1 Ganea's problems

Problems [T. Ganea, 1971, (15 problems)]

1. Compute $\operatorname{cat} M$ for a manifold M.

2. $\operatorname{cat} X \times S^n = \operatorname{cat} X + 1$. Is it true?

4. Let E be the total space of a sphere bundle over a sphere. Describe $\operatorname{cat} E$ in terms of homotopy invariants of the characteristic map of E.

10. Is a co-H-space X homotopy equivalent to a wedge of a simply-connected space and circles?

Remark 1.1 *According to the James' handbook on algebraic topology, the affirmative answers to*

Problems 2 (LS category) and 10 (co-H-spaces) are supposed to be true and are called "the Ganea conjecture" in each area.

2 Lusternik-Schnirelmann category

Definition 2.1

$$\mathrm{cat}\,(X) = \mathrm{Min}\left\{ m \middle| \begin{array}{c} \exists\{U_0, ..., U_m : \text{ open in } X\} \\ X = \displaystyle\bigcup_{i=0}^{m} U_i, \ \begin{array}{l}\text{each } U_i \text{ is con-} \\ \text{tractible } \underline{in}\ X\end{array} \end{array} \right\}$$

A topological invariant $g\mathrm{cat}\,(X)$ *is defined similarly but is not a homotopy invariant (R. H. Fox)*

$$g\mathrm{cat}\,(X) = \mathrm{Min}\left\{ m \middle| \begin{array}{c} \exists\{U_0, ..., U_m : \text{ open in } X\} \\ X = \displaystyle\bigcup_{i=0}^{m} U_i, \ \begin{array}{l}\text{each } U_i \text{ is con-} \\ \text{tractible}\end{array} \end{array} \right\}$$

$$\mathrm{Cat}\,(X) = \mathrm{Min}\left\{ m \middle| \exists_{\{Y(\simeq X)\}}\ g\mathrm{cat}\,(Y) = m \right\}$$

Theorem 2.2 (Lusternik-Schnirelmann)

The number of critical points of any C^∞ map on a manifold M is greater than $\operatorname{cat} M$.

Theorem 2.3 (Ganea 1971)

$$\operatorname{Cat} X - 1 \leq \operatorname{cat} X \leq \operatorname{Cat} X \leq g\operatorname{cat} X.$$

So, there are two cats homotopy-theoretically, small and big. In fact, there is a lots of new variants of cats, like $w\operatorname{cat}$, $\sigma\operatorname{cat}$, cl, and their rational verisions, local versions, etc.

But we know the two oldest cats cat and Cat are the strongest.

3 A_∞ structure

For a space X, its loop space ΩX has an A_∞-structure, i.e, there is a ladder of quasi-fibrations $\{p_m^{\Omega X}\}$.

$$
\begin{array}{ccccccccccccc}
\Omega X & \overset{*}{\underset{\wr}{\hookrightarrow}} & E^2(\Omega X) & \overset{*}{\underset{\wr}{\hookrightarrow}} & \cdots & \overset{*}{\underset{\wr}{\hookrightarrow}} & E^m(\Omega X) & \overset{*}{\underset{\wr}{\hookrightarrow}} & E^{m+1}(\Omega X) & \overset{*}{\underset{\wr}{\hookrightarrow}} & \cdots & \overset{*}{\underset{\wr}{\hookrightarrow}} & E^\infty(\Omega X) \\
\downarrow{\scriptstyle p_1^{\Omega X}} & & \downarrow{\scriptstyle p_2^{\Omega X}} & & & & {\scriptstyle p_m^{\Omega X}}\downarrow & & {\scriptstyle p_{m+1}^{\Omega X}}\downarrow & & & & {\scriptstyle p_\infty^{\Omega X}}\downarrow \\
\{*\} & \hookrightarrow & P^1(\Omega X) & \hookrightarrow & \cdots & \hookrightarrow & P^{m-1}(\Omega X) & \hookrightarrow & P^m(\Omega X) & \hookrightarrow & \cdots & \hookrightarrow & P^\infty(\Omega X) \\
 & & & & & & & & & & & & \downarrow\wr \\
 & & & & & & & & & & & & X
\end{array}
$$

The existence of these kind of ladders is equivalent with the existence of the higher homotopy associativity $\{M_m^{\Omega X}\}_{m\geq 1}$ for the loop space ΩX. The ladder derived from the canonical higher homotopy $\{M_m^{\Omega X}\}_{m\geq 1}$ enjoys a kind of universality (Stasheff 1963).

Theorem 3.1 *For a space X, $\operatorname{cat} X \leq m$ iff there is a homotopy cross-section $\sigma(X) : X \to P^m(\Omega X)$ of $e_m^{\Omega X} : P^m(\Omega X) \hookrightarrow P^\infty(\Omega X) \simeq X$.*

We call this $\sigma(X)$ the structure map for $\operatorname{cat} X \leq m$.

Definition 3.2 *For a nilpotent space X, $\operatorname{cat}_p X \leq m$ iff there is a map $\sigma : X \to P^m(\Omega X)$ such that $e_m^{\Omega X} \circ \sigma : X \to X$ is a homotopy equivalence.*

Stasheff's A_∞-form yields the following result.

Theorem 3.3 *For any spaces X and Y, $\operatorname{cat} X \times Y \leq m$ iff there is a homotopy cross-section $\sigma(X \times Y) : X \times Y \to \bigcup_{i+j=m} P^i(\Omega X) \times P^j(\Omega Y)$ of $e_m^{\Omega X} \times e_m^{\Omega Y}$.*

4 Problem 2 (the Ganea conjecture on LS category)

The Hess-Jessup method on rational homotopy theory proves the rational version of the conjecture.

Theorem 4.1 (Hess 1991, Jessup 1990)

$$\operatorname{cat}_0 X \times S^n = \operatorname{cat}_0 X + 1, \quad n \geq 2,$$

where cat_0 *denotes the rationalisation of* cat *.*

For a manifolds, Rudyak improves a result of Singhof.

Theorem 4.2 (Rudyak 1997, Singhof 1979)

For a large class of manifolds M*,* $\operatorname{cat} M \times S^n = \operatorname{cat} M + 1$*,* $n \geq 2$

The following results were obtained using higher Hopf invariants defined on projective spaces associated with Stasheff's A_∞-structure of a loop space.

4.1 (integral case)

Let V be a $(d-1)$-connected co-H-space and X a $(d-1)$-connected complex, $d \geq 2$ with $\operatorname{cat} X = m$.

Theorem 4.3 *Let X be of $\dim X \leq d \cdot \operatorname{cat} X + d - 2$ and $n \geq 1$. Then the following statement holds for $W = X \cup_f C(V)$ $(f : V \to X)$.*

$$\operatorname{cat} W = \operatorname{cat} X + 1 \quad \textit{iff} \quad H_m^{\sigma(X)}(f) \neq 0.$$

Theorem 4.4 *Under the same conditions as in Theorem 4.3, the following equation holds for $W =$*

$X \cup_f C(V)$ *($f : V \to X$), when* $\operatorname{cat} W = \operatorname{cat} X + 1$.

$$\operatorname{cat} W \times S^n = \operatorname{cat} W + 1 \quad \textit{iff} \quad \Sigma^n H_m^{\sigma(X)}(f) \neq 0.$$

Using Toda's result (1957,1962) on the non-existence of elements of Hopf invariant one in $\pi_{31}(S^{16})$, we obtain the following result.

Theorem 4.5 (I. 1998) *There is a space Q such that* $\operatorname{cat}(Q \times S^k) = \operatorname{cat} Q = 2$, *for any $k \geq 1$.*

Theorem 4.6 (I. 1998) *There is a series of spaces $Q(p, m, 2n)$ for any odd primes p and integers m, n such that* $\operatorname{cat}(Q(p, m, 2n)) = m$ *and*

$$\operatorname{cat}(Q(p, m, 2n) \times S^k) = \begin{cases} m + 1, & k < 2n \\ m, & k \geq 2n. \end{cases}$$

4.2 (p-local case)

Theorem 4.7 *For $k \geq 1$ and an odd prime p,*

$$\mathrm{cat}_2\,(Q\times S^k) = \mathrm{cat}_2\,Q = 2,$$

$$\mathrm{cat}_p\,(Q\times S^k) = 2 \text{ and } \mathrm{cat}_p\,Q = 1, \qquad .$$

Theorem 4.8 *For $k \geq 2n$ and a prime $q \neq p$,*

$$\mathrm{cat}_p\,(Q(p,m,2n)\times S^k) = \mathrm{cat}_p\,Q(p,m,2n) = m,$$

$$\mathrm{cat}_q\,(Q(p,m,2n)\times S^k) = m = \mathrm{cat}_q\,Q(p,m,2n) + 1.$$

Thus we also have many counter examples to the Ganea conjecture on cat_p.

4.3 (rational case)

Let V be a $(d-1)$-connected co-H-space and X a $(d-1)$-connected complex, $d \geq 2$ with $\mathrm{cat}_0\,X = m$.

Theorem 4.9 *Let X be of $\dim X \leq d \cdot \mathrm{cat}_0 X + d - 2$ and $n \geq 1$. Then for $W = X \cup_f C(V)$, where $f : V \to X$, the following equation holds.*

$$\mathrm{cat}_0 W \times S^1 = \mathrm{cat}_0 W + 1.$$

This gives a positive partial answer to the Ganea conjecture on cat_0 for $n = 1$.

5 Problem 4

Let $r \geq 1$, $q \geq 1$ and E be a bundle over S^{q+1} with fibre S^{r+1}. Then $E \simeq S^{r+1} \cup_\alpha e^{q+1} \cup_\psi e^{q+r+2}$ with attaching maps $\alpha : S^q \to S^{r+1}$ and $\psi : S^{q+r+1} \to Q = S^{r+1} \cup_\alpha e^{q+1}$.

Fact 5.1 *Let* $\alpha = 1_{S^{r+1}}$ *the identity. Then clearly* $\operatorname{cat} Q = 0$ *and* $\operatorname{cat} E = 1$*. In addition,* $\operatorname{cat} Q \times S^n = 1$ *and* $\operatorname{cat} E \times S^n = 2$ *for* $n \geq 1$*.*

Fact 5.2 *Let* $\alpha \neq 1_{S^{r+1}}$*. Hence* $1 \leq \operatorname{cat} Q \leq 2$*. Then* $\operatorname{cat} Q = 2$ *if and only if* $H_1(\alpha) \neq 0$*. In particular if* $H_1(\alpha) = 0$*, we can easily obtain that* $\operatorname{cat} Q = 1$ *and* $\operatorname{cat} E = 2$*. In this case, it also follows that* $\operatorname{cat} Q \times S^n = 2$ *and* $\operatorname{cat} E \times S^n = 3$ *for* $n \geq 1$*.*

The method given in the previous section allow us to compute further.

Theorem 5.3 *Let* $H_1(\alpha) \neq 0$*. Hence* $\operatorname{cat} Q = 2$*. Then for* $n \geq 1$*,* $\operatorname{cat} Q \times S^n = 3$ *if and only if*

$\Sigma^n H_1(\alpha) \neq 0$.

Theorem 5.4 *Let $H_1(\alpha) \neq 0$. Hence we have $2 \leq \operatorname{cat} E \leq 3$. We have $\operatorname{cat} E = 3$ if $\Sigma^{r+2} h_2(\alpha) \neq 0$. Also we have $\operatorname{cat} E = 2$ if $H_2(\psi) = 0$ for some choice of $\sigma(Q) : Q \to P^2(\Omega Q)$.*

Theorem 5.5 *Let $\Sigma^{r+2} h_2(\alpha) \neq 0$. Hence we have $\operatorname{cat} E = 3$. We have for $n \geq 1$, $\operatorname{cat} E \times S^n = 4$ if $\Sigma^{n+r+2} h_2(\alpha) \neq 0$. Also we have $\operatorname{cat} E \times S^n = 3$ if $\Sigma^n H_2(\psi) = 0$ for some choice of $\sigma(Q) : Q \to P^2(\Omega Q)$.*

Using Oka's results on p-primary components of $\pi_*^S(S^0)$, we obtain the following result.

Theorem 5.6 *Let p be an odd prime, β be the co-H-map $\alpha_1(3) : S^{2p} \to S^3$ and γ be the suspension map $\alpha_2(2p) = \Sigma^{2p-3}\alpha_2(3) : S^{6p-5} \to S^{2p}$ for the prime p. Then $\Sigma H_2(\psi(\beta\circ\gamma))$ is the composition of a map $\pm\Sigma^3(\beta\circ\gamma)$ with an appropriate inclusion map.*

6 co-H-space

Fact 6.1 *For a finite Hopf space X (e.g. a compact Lie group), there is a homotopy equivalence $X \simeq S^1\times\cdots\times S^1\times D$ with $H^1(D) = 0$.*

Dualising this, we can show the following result.

Theorem 6.2 (Oda,I.) *For a co-H-space X (e.g,* *a suspension space), there is a homology equivalence* $X \to S^1 \vee \cdots \vee S^1 \vee D$ *with* $\pi_1(D) = 0$ *which also induces an isomorphism of fundamental groups.*

7 Problem 10 (the Ganea conjecture on a co-H-space)

Definition 7.1 *A space* X *is "standard" iff there is a homotopy equivalence* $X \simeq S^1 \vee \cdots \vee S^1 \vee D$ *with* $\pi_1(D) = 0$.

Problem 10 was studied in 70's by several authors, e.g, Berstein-Dror (1976), Hilton-Mislin-Roitberg (1978), using the given *co-H-structure* itself on a co-H-space.

Fact 7.2 *For a co-H-space X, Ganea's condition 1) is equivalent with the conditions 2) to 5) below.*

1) *(Ganea) X is "standard".*

2) *(Berstein-Dror) The co-action of B along $j : X \to B$ associated with the given co-H-structure of X can be chosen as associative.*

3) *(Hilton-Mislin-Roitberg) The co-H-structure of X can be chosen to make the left (or right) co-shear map a homotopy equivalence.*

4) *(Hilton-Mislin-Roitberg) The co-H-structure of X can be chosen to be co-loop, i.e, it induces a natural algebraic-loop structure on $[X, -]$.*

5) (Hilton-Mislin-Roitberg) The co-H-structure of X can be chosen to make $e = i \circ j$ loop-like from the left (or right).

Contrary to the above, some authors have obtained results not depending on the co-H-structure itself.

Theorem 7.3 (Henn 1983) *An almost rational co-H-space X is "standard": $X \simeq S^1 \vee \cdots \vee S^1 \vee \bigvee_i S^{n_i}_{(0)}$ with $n_i \geq 2$.*

So the rational version of the Ganea conjecture on a co-H-space is true.

Definition 7.4 *A space X is of (almost) stable dimension $\leq k$, iff the homology of $\widetilde{X}$ is concen-*

trated in $H_{n+1}, \ldots, H_{n+k}$ *for some* $n \geq 0$ *with* H_{n+k} *torsion free.*

Theorem 7.5 (Komatsu 1992) *Let* X *be the exterior of a boundary link. If* X *is a co-H-space (of stable dimension 1), then* X *is "standard".*

Komatsu showed this using Fox's free differential calculous.

Theorem 7.6 (Saito-Sumi-I. 1998) *Let* X *be of stable dimension* ≤ 2*. If* X *is a co-H-space, then* X *is "standard".*

The main tool to show this is the following result.

Proposition 7.7 *If X is a co-H-space, then there is the following commutative diagram:*

$$\begin{array}{ccc} H_*(\widetilde{X},\widetilde{B}) & \xrightarrow{\cong} & \mathbb{Z}\pi\otimes H_*(X,B) \\ {\scriptstyle p(X)_*}\downarrow & \text{commutative} & \downarrow{\scriptstyle \mathbb{Z}\otimes_{\mathbb{Z}\pi}(-)} \\ H_*(X,B) & = & H_*(X,B), \end{array} \tag{7.1}$$

where $\pi = \pi_1(X)$.

This is obtained by the following lemma shown by using Bass' proof of $K(\mathbb{Z}\pi) = 0$ on algebraic K-theory.

Lemma 7.8 *If a $\mathbb{Z}\pi$-module P is a direct summand of $\mathbb{Z}\pi \otimes M$ for some module M, then $P \cong \mathbb{Z}\pi \otimes P_0$ as $\mathbb{Z}\pi$-modules for some module P_0.*

While there are only 2-torsions up to 2-stem, we know $\pi_3^S(S^0) \cong \mathbb{Z}/24\mathbb{Z}$, $24 = 2^3 \cdot 3$. This causes a problem to showing the Ganea conjecture on a co-H-space. And a series of complexes is eventually found.

Theorem 7.9 (I. 1999) *There is a series of co-H-spaces* $\{R_n = (S^1 \vee S^{n+1}) \cup e^{n+5} \,|\, n \geq 4\}$ *satisfying the following properties.*

1) The almost p-localisation of R_n *is standard for any prime p.*

2) The almost rationalisation of R_n *is standard.*

3) $\pi_*(R_n) \cong \pi_*(S^1 \vee (S^{n+1} \cup e^{n+5}))$.

4) R_n *is not standard.*

[proof] We know that $\pi_{n+4}(S^{n+1}) \cong \mathbb{Z}/24\mathbb{Z}\{\nu_{n+1}\}$, $n \geq 4$. Since $24 = 2^3 \times 3$, $C_n = S^{n+1} \cup_{\nu_{n+1}} e^{n+5}$ does

not split into a wedge sum of spheres at primes 2 and 3. We define $R_n = (S^1 \vee S^{n+1}) \cup_\Psi e^{n+5}$ to satisfy

$$\tilde{H}_*(\widetilde{R_n}; \mathbb{Z}) \cong \mathbb{Z}\pi\{x_{n+1}, x_{n+5}\},$$

$$\tilde{H}_*(\widetilde{R_n}; \mathbb{F}_2) \cong \mathbb{F}_2\pi\{x'_{n+1}, x'_{n+5}\},$$

$$\tilde{H}_*(\widetilde{R_n}; \mathbb{F}_3) \cong \mathbb{F}_3\pi\{x''_{n+1}, x''_{n+5}\},$$

$$x'_{n+5}\mathcal{S}q^4 = x'_{n+1}, \quad \text{and} \quad x''_{n+5}\mathcal{P}^1 = \tau\cdot x''_{n+1},$$

where τ is the generator of $\pi \cong \mathbb{Z}$. On the other hand, the following is clear.

$$\tilde{H}_*(\widetilde{S^1 \vee C_n}; \mathbb{Z}) \cong \mathbb{Z}\pi\{u_{n+1}, u_{n+5}\},$$

$$\tilde{H}_*(\widetilde{S^1 \vee C_n}; \mathbb{F}_2) \cong \mathbb{F}_2\pi\{u'_{n+1}, u'_{n+5}\},$$

$$\tilde{H}_*(\widetilde{S^1 \vee C_n}; \mathbb{F}_3) \cong \mathbb{F}_3\pi\{u''_{n+1}, u''_{n+5}\},$$

$$u'_{n+5}\mathcal{S}q^4 = u'_{n+1}, \quad \text{and} \quad u''_{n+5}\mathcal{P}^1 = u''_{n+1}.$$

Then one can easily see, at each prime, there is a homotopy equivalence from R_n to $S^1 \vee C_n$, because the homomorphism multiplying 1 or τ is an isomorphism. The key to show that R_n is not standard is as follows:

Key Lemma 7.10 *The set of invertible elements in the group ring* $\mathbb{Z}\pi$ *is* $\pm\pi \subset \mathbb{Z}\pi$*.*

If a homotopy equivalence $f : R_n \to S^1 \vee C_n$ exists, it induces the $\mathbb{Z}\pi$-module isomorphisms such that

$$f_* x_{n+1} = \pm\tau^i u_{n+1}$$

$$f_* x_{n+5} = \pm\tau^j u_{n+5}$$

Reducing modulo 2 and 3, we have $i = j$ and $i = j-1$. It's a contradiction.

To show that R_n is a co-H-space, we use a characterisation of a space with co-action given in [Saito-Sumi-I.]. *QED.*

This might suggest the following conjecture.

Conjecture 1 *For any co-H-space X, the following are always true.*

1) The almost p-localisation of X is standard for any prime p.

2) $\pi_*(X)$ is isomorphic with $\pi_*(B \vee C)$, for $B = B\pi_1(X)$ and $C = X/B$.